Reverse Orthogonal Polynomials

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Let us begin by recalling the Gram-Schmidt procedure.

Let V be a real (or complex) vector space with a (positive-definite) inner product. For example, we could let $V = \mathbb{R}^n$ (or $\mathbb{C}^n$) with inner product the usual "dot product", i.e., $\langle v, w \rangle = v \cdot w$ (or $\langle v, w \rangle = v \cdot \overline{w}$).

Theorem. *Let W be a subspace of V of finite or countably infinite dimension, and let $\mathscr{B} = \{x_1, x_2, \dots\}$ be an arbitrary ordered basis of W. Let $\mathscr{C} = \{y_1, y_2, \dots\}$ and $\mathscr{D} = \{z_1, z_2, \dots\}$ be defined as follows:*

Let $y_1 = x_1$ and $z_1 = y_1 / \|y_1\|$.

If $y_1, \dots, y_{i-1}$ are defined, let

$$y_i = x_i - \sum_{k=1}^{i-1} (\langle x_i, y_k \rangle / \|y_k\|^2) y_k \text{ and } z_i = y_i / \|y_i\|.$$

Then $\mathscr{C}$ is an orthogonal basis of W and $\mathscr{D}$ is an orthonormal basis of W. Furthermore, $\mathrm{Span}(\{z_1, \dots, z_i\}) = \mathrm{Span}(\{y_1, \dots, y_i\}) = \mathrm{Span}(\{x_1, \dots, x_i\})$ *for each i.*

The bases $\mathscr{C}$ and $\mathscr{D}$ obtained in this way are said to be obtained from the basis $\mathscr{B}$ by the Gram-Schmidt procedure.

Proof. Suppose that $\{y_1,\ldots,y_{i-1}\}$ is orthogonal. Then for any j between 1 and $i-1$,

$$\begin{aligned}\langle y_i,y_j\rangle &= \langle x_i - \sum_{k=1}^{i-1}(\langle x_i,y_k\rangle/\|y_k\|^2)y_k, y_j\rangle \\ &= \langle x_i,y_j\rangle - \sum_{k=1}^{i-1}\langle x_i,y_k\rangle/\|y_k\|^2\langle y_k,y_j\rangle \\ &= \langle x_i,y_j\rangle - (\langle x_i,y_j\rangle/\|y_j\|^2)(\|y_j\|^2) = 0\end{aligned}$$

as $\langle y_k,y_j\rangle = 0$ for $k \neq j$ by orthogonality, and as $\langle y_j,y_j\rangle = \|y_j\|^2$. Hence $\{y_1,\ldots,y_i\}$ is orthogonal.

Thus by induction the set $\{y_1,\ldots,y_i\}$ is orthogonal for any i, and that means, even if there are infinitely many vectors, that the set $\{y_1,y_2,\ldots\}$ is orthogonal, as given any two vectors y_j and y_k in this set they lie in some finite subset, namely in $\{y_1,\ldots,y_i\}$ with $i = \max(j,k)$.

Then if $\{y_1,y_2,\ldots\}$ is orthogonal, $\{z_1,z_2,\ldots\}$ is orthonormal, as each z_i is a multiple of y_i and $\|z_i\| = 1$ for each i.

Finally, from the construction of y_i we see that y_i is of the form $y_i = x_i + c_1y_1 + \cdots + c_{i-1}y_{i-1}$ for some scalars $c_1,\ldots,c_{i-1}$, for each i, so $\mathrm{Span}(\{z_1,\ldots,z_i\}) = \mathrm{Span}(\{y_1,\ldots,y_i\}) = \mathrm{Span}(\{x_1,\ldots,x_i\})$ for each i. □

Note that the result of the Gram-Schmidt procedure depends on the order of the vectors in the original ordered basis $\mathscr{B}$.

Example. Let $V = \mathbb{R}^3$ and let $\mathscr{B} = \{\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\4\end{bmatrix}\}$. We use the Gram-Schmidt procedure to convert $\mathscr{B}$ to an orthonormal basis $\mathscr{D}$ of $\mathbb{R}^3$.

We begin with $x_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$. Then $y_1 = x_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$ and $z_1 = y_1/\|y_1\| = \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$.

Next consider $x_2 = \begin{bmatrix}0\\2\\3\end{bmatrix}$. Then $y_2 = x_2 - (\langle x_2,y_1\rangle/\|y_1\|^2)y_1$,

$$y_2 = \begin{bmatrix}0\\2\\3\end{bmatrix} - \left(\langle \begin{bmatrix}0\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\rangle/9\right)\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}0\\2\\3\end{bmatrix} - (10/9)\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}-10/9\\-2/9\\7/9\end{bmatrix}$$

and $z_2 = y_2/\|y_2\| = \begin{bmatrix}-10/\sqrt{17}\\-2/3\sqrt{17}\\7/3\sqrt{17}\end{bmatrix}$.

Next consider $x_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. Then $y_3 = x_3 - (\langle x_3, y_1 \rangle / \|y_1\|^2) y_1 - (\langle x_3, y_2 \rangle / \|y_2\|^2) y_2$,

$$y_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \left(\langle \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \rangle / 9 \right) \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left(\langle \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -10/9 \\ -2/9 \\ 7/9 \end{bmatrix} \rangle / (17/9) \right) \begin{bmatrix} -10/9 \\ -2/9 \\ 7/9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - (14/9) \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - (22/17) \begin{bmatrix} -10/9 \\ -2/9 \\ 7/9 \end{bmatrix} = \begin{bmatrix} -2/17 \\ 3/17 \\ -2/17 \end{bmatrix}$$

and $z_3 = y_3 / \|y_3\| = \begin{bmatrix} -2/\sqrt{17} \\ 3/\sqrt{17} \\ -2/\sqrt{17} \end{bmatrix}$.

Example. Let $V = \mathbb{R}^3$ and let $\mathscr{B} = \{ \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \}$. We use the Gram-Schmidt procedure to convert $\mathscr{B}$ to an orthonormal basis $\mathscr{D}$ of $\mathbb{R}^3$.

We begin with $x_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. Then $y_1 = x_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ and $z_1 = y_1 / \|y_1\| = \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}$.

Next consider $x_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$. Then $y_2 = x_2 - (\langle x_2, y_1 \rangle / \|y_1\|^2) y_1$,

$$y_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \left(\langle \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \rangle / 25 \right) \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - (18/25) \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4/25 \\ 3/25 \end{bmatrix}$$

and $z_2 = y_2 / \|y_2\| = \begin{bmatrix} 0 \\ -4/5 \\ 3/5 \end{bmatrix}$.

Next consider $x_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Then $y_3 = x_3 - (\langle x_3, y_1 \rangle / \|y_1\|^2) y_1 - (\langle x_3, y_2 \rangle / \|y_2\|^2) y_2$,

$$y_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left(\langle \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \rangle / 25 \right) \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \left(\langle \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -4/25 \\ 3/25 \end{bmatrix} \rangle / (1/25) \right) \begin{bmatrix} 0 \\ -4/25 \\ 3/25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - (14/25) \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - (-2) \begin{bmatrix} 0 \\ -4/25 \\ 3/25 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and $z_3 = y_3 / \|y_3\| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Now we look at the situation that interests us.

Let $\mathscr{P} = \{$real polynomials $f(x)\}$ with inner product

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,dx$$

for a suitable "weight" function $w(x)$.

Let $\mathscr{B} = \{1, x, x^2, \ldots\}$. We apply the Gram-Schmidt procedure to obtain an orthogonal basis $\{F_0(x), F_1(x), F_2(x), \ldots\}$ for $\mathscr{P}$, except we normalize by specifying the values $\{F_n(1) \mid n = 0, 1, \ldots\}$.

Here are three classical cases:

(a) $w(x) = 1$ and $F_n(1) = 1$ for every n. The polynomials obtained in this way are the Legendre polynomials $\{P_n(x)\}$. The first few of these polynomials are given by

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)
\end{aligned}$$

(b) $w(x) = 1/\sqrt{1-x^2}$ and $F_n(1) = 1$ for every n. The polynomials obtained in this way are the Chebyshev polynomials of the first kind $\{T_n(x)\}$. The first few of these polynomials are given by

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= (2x^2 - 1) \\
T_3(x) &= (4x^3 - 3x) \\
T_4(x) &= (8x^4 - 8x^2 + 1) \\
T_5(x) &= (16x^5 - 20x^3 + 5x) \\
T_6(x) &= (32x^6 - 48x^4 + 18x^2 - 1) \\
T_7(x) &= (64x^7 - 112x^5 + 56x^3 - 7x)
\end{aligned}$$

(c) $w(x) = \sqrt{1-x^2}$ and $F_n(1) = n+1$ for every n. The polynomials obtained in this way are the Chebyshev polynomials of the second kind $\{U_n(x)\}$. The first few of these polynomials are given by

$$
\begin{aligned}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_2(x) &= (4x^2 - 1) \\
U_3(x) &= (8x^3 - 4x) \\
U_4(x) &= (16x^4 - 12x^2 + 1) \\
U_5(x) &= (32x^5 - 32x^3 + 6x) \\
U_6(x) &= (64x^6 - 80x^4 + 24x^2 - 1) \\
U_7(x) &= (128x^7 - 192x^5 + 80x^3 - 8x)
\end{aligned}
$$

These polynomials are all known in general.

We think of these polynomials as being obtained by applying the Gram-Schmidt procedure "from the bottom up", and it seemed to us a natural question to ask what happens if we apply this procedure "from the top down". Of course, $\mathscr{B}$ has no "top". But instead, for any nonnegative integer n, we may let $\mathscr{P}_n$ be the vector space of polynomials of degree at most n, equipped with the same inner product, and apply the Gram-Schmidt procedure to the ordered basis $\mathscr{B} = \{x^n, x^{n-1}, \ldots, 1\}$ of $\mathscr{P}_n$ to obtain polynomials

$$\{\overleftarrow{F}^n_n(x), \overleftarrow{F}^n_{n-1}(x), \ldots, \overleftarrow{F}^n_0(x)\}$$

with appropriate normalization.

Since these polynomials are obtained by reversing the order of the basis elements, we call these reverse orthogonal polynomials. In particular we have:

(a) $w(x) = 1$ and $\overleftarrow{P}^n_k(1) = 1$ for every n, k, the reverse Legendre polynomials.

(b) $w(x) = 1/\sqrt{1-x^2}$ and $\overleftarrow{T}^n_k(1) = 1$ for every n, k, the reverse Chebyshev polynomials of the first kind.

(b) $w(x) = \sqrt{1-x^2}$ and $\overleftarrow{U}^{2m+1+k}_k(1) = \overleftarrow{U}^{2m+k}_k(1) = 2m+2$ for every m, k, the reverse Chebyshev polynomials of the second kind.

These are our objects of interest.

We explicitly determine these polynomials and describe some of their properties.

General properties of reverse orthogonal polynomials

We begin more generally, but we make the restriction that the weight function $w(x)$ is an *even* function.

We let $\{c_k^n\}$ be an arbitrary set of nonzero real numbers and normalize the functions $\{\overleftarrow{F}_k^n(x)\}$ by setting $\overleftarrow{F}_k^n(1) = c_k^n$.

Theorem 1. *Let m, n and k be nonnegative integers with $k \leq n$.*

(a) The reverse orthogonal polynomial $\overleftarrow{F}_k^n(x)$ is a polynomial of degree at most n whose low-order term is a nonzero multiple of x^k.

(b) $\overleftarrow{F}_k^n(x)$ is an even polynomial if k is even and an odd polynomial if k is odd.

(c) $\overleftarrow{F}_k^{2m+k+1}(x) = (c_k^{2m+1+k}/c_k^{2m+k})\overleftarrow{F}_k^{2m+k}(x)$. In particular, if $c_k^{2m+1+k} = c_k^{2m+k}$, $\overleftarrow{F}_k^{2m+k+1}(x) = \overleftarrow{F}_k^{2m+k}(x)$.

(d) $\overleftarrow{F}_k^{2m+k}(x)$ is a polynomial of degree $2m+k$, with a zero of order k at $x = 0$ and $2m$ simple zeroes at nonzero values of x, all of them real numbers symmetrically located around the origin and lying in the open interval $(-1,1)$.

(e) $\overleftarrow{F}_k^n(x)$ is a polynomial of degree n if $n-k$ is even and of degree $n-1$ if $n-k$ is odd.

(f) The reverse orthogonal polynomial $\overleftarrow{F}_k^n(x)$ is uniquely determined by the condition (a) above and by the orthogonality relations $\langle \overleftarrow{F}_j^n(x), \overleftarrow{F}_k^n(x)\rangle = 0$ for $j > k$ and the normalization $\overleftarrow{F}_k^n(1) = c_0^n$.

Proof. (d) By (a) and (b), we have that $\overleftarrow{F}_k^{2m+k}(x) = x^k f(x)$ for some even polynomial $f(x)$ of degree at most $2m$ with nonzero constant term. Thus the zeroes of $f(x)$ all occur at nonzero values of x, symmetrically located with respect to the origin. Let $f(x)$ have t zeroes of odd order $r_1, \ldots, r_t$ in the interval $(-1,1)$. We have that $t \leq 2m$, so if we show that $t = 2m$ we will have established that these are all the zeroes of $f(x)$, that they are all simple, and that $\overleftarrow{F}_k^{2m+k}(x)$ has degree $2m+k$. We argue by contradiction. Suppose $t < 2m$. Since t is even, $t \leq 2m-2$. Let $g(x) = (x-r_1)\cdots(x-r_t)$ and $h(x) = x^{k+2}g(x)$. Then on the one hand

$$\langle \overleftarrow{F}_k^{2m+k}(x), h(x)\rangle = \int_{-1}^{1} (x^k f(x))(x^{k+2}g(x))w(x)dx = \int_{-1}^{1} x^{2k+2} f(x)g(x)w(x)dx \neq 0,$$

as $x^{2k+2}f(x)g(x)$ has constant sign in $[-1,1]$ and is not identically zero. But on the other hand, $h(x)$ is a polynomial of degree at most $(k+2)+(2m-2) = 2m+k$ with low-order term of degree $k+2$, so $\overleftarrow{F}_k^{2m+k}(x)$ is orthogonal to $h(x)$, i.e., $\langle \overleftarrow{F}_k^{2m+k}(x), h(x)\rangle = 0$; contradiction.

(f) If V is the vector space of polynomials whose high-order term has degree at most n and whose low-order term has degree at least k, and U is the subspace of polynomials whose high-order term has degree at most n and whose low-order term has degree at least $k+1$, then U is a codimension 1 subspace of V so its orthogonal complement is 1-dimensional, consisting of multiples of any nonzero element. $\overleftarrow{F}_k^n(x)$ is an element of this subspace, and so to specify it uniquely we need only normalize it, and we do so by specifying $\overleftarrow{F}_k^n(1)$. (In order to do so, we must know that no nonzero element of this subspace is 0 for $x=1$, and that follows from (d).) □

There are some cases in which we can readily determine $\overleftarrow{F}_k^n(x)$.

Theorem 2. *(a)* $\overleftarrow{F}_n^{n+1}(x) = c_n^{n+1}x^n$ *and* $\overleftarrow{F}_n^n(x) = c_n^n x^n$ *for all* $n \geq 0$.
In particular, if $c_n^{n+1} = c_n^n$, $\overleftarrow{F}_n^{n+1}(x) = \overleftarrow{F}_n^n(x)$.

(b) $\overleftarrow{F}_0^{2m+1}(x) = (c_0^{2m+1}/F_{2m+1}(1))F_{2m+1}(x)/x$ *and* $\overleftarrow{F}_0^{2m} = (c_0^{2m}/F_{2m+1}(1))F_{2m+1}(x)/x$ *for all* $m \geq 0$.
In particular, if $c_0^{2m+1} = c_0^{2m} = F_{2m+1}(1)$, $\overleftarrow{F}_0^{2m+1}(x) = \overleftarrow{F}_0^{2m}(x) = F_{2m+1}(x)/x$.

Proof. (a) is immediate.

As for (b), let $n = 2m$ or $2m+1$. We first note that $F_{2m+1}(x)$ is an odd polynomial of degree $2m+1$ with nonzero x term, so the quotient $F_{2m+1}(x)/x$ is an even polynomial of degree $2m \leq n$ with a nonzero constant term. We show that $F_{2m+1}(x)/x$ satisfies the conditions of Theorem 1.3(f) and therefore $\overleftarrow{F}_0^{2m+1}(x) = F_{2m+1}(x)/x$. By Theorem 1.3(a), we have that, for any j between 1 and n, $\overleftarrow{F}_j^n(x)$ is a polynomial of degree at most n that is divisible by x^j, so in particular, for any such j, $\overleftarrow{F}_j^n(x)/x$ is a polynomial of degree at most $n-1 \leq 2m$. But then

$$\begin{aligned}\langle \overleftarrow{F}_j^n(x), F_{2m+1}(x)/x\rangle &= \int_{-1}^{1} \overleftarrow{F}_j^n(x)(F_{2m+1}(x)/x)w(x)dx \\ &= \int_{-1}^{1} (\overleftarrow{F}_j^n(x)/x)F_{2m+1}(x)w(x)dx = \langle \overleftarrow{F}_j^n(x)/x, F_{2m+1}(x)\rangle = 0,\end{aligned}$$

as $F_{2m+1}(x)$ is orthogonal to every polynomial of degree at most $2m$.

Also, the value of the polynomial $\overleftarrow{F}_{2m+1}(x)$ at $x=1$ is $(c_0^{2m+1}/F_{2m+1}(1))F_{2m+1}(1)/1 = c_0^{2m+1}$. and the value of the polynomial $\overleftarrow{F}_{2m}(x)$ at $x=1$ is $(c_0^{2m}/F_{2m+1}(1))F_{2m+1}(1)/1 = c_0^{2m}$. □

In particular, we have:

(a) $\overleftarrow{P}_n^n(x) = x^n$ and $\overleftarrow{P}_0^{2m+1}(x) = \overleftarrow{P}_0^{2m}(x) = P_{2m+1}(x)/x$.

(b) $\overleftarrow{T}_n^n(x) = x^n$ and $\overleftarrow{T}_0^{2m+1}(x) = \overleftarrow{T}_0^{2m}(x) = T_{2m+1}(x)/x$.

(a) $\overleftarrow{U}_n^n(x) = 2x^n$ and $\overleftarrow{U}_0^{2m+1}(x) = \overleftarrow{U}_0^{2m}(x) = U_{2m+1}(x)/x$.

Tables of values

Let $J_k^n = 2^{j(n-k)}$ where $j(0) = j(1) = 0,\ j(2) = j(3) = 1,\ j(4) = j(5) = 3,\ j(6) = j(7) = 4.$

Table of $J_k^n \overleftarrow{P}_k^n(x)$

6							x^6
5						x^5	x^5
4					x^4	x^4	$13x^6-11x^4$
3				x^3	x^3	$11x^5-9x^3$	$11x^5-9x^3$
2			x^2	x^2	$9x^4-7x^2$	$9x^4-7x^2$	$143x^6-198x^4+63x^2$
1		x	x	$7x^3-5x$	$7x^3-5x$	$99x^5-126x^3+35x$	$99x^5-126x^3+35x$
0	1	1	$5x^2-3$	$5x^2-3$	$63x^4-70x^2+15$	$63x^4-70x^2+15$	$429x^6-693x^4+315x^2-35$
	0	1	2	3	4	5	6

Table of $\overleftarrow{T}_k^n(x)$

6							x^6
5						x^5	x^5
4					x^4	x^4	$12x^6-11x^4$
3				x^3	x^3	$10x^5-9x^3$	$10x^5-9x^3$
2			x^2	x^2	$8x^4-7x^2$	$8x^4-7x^2$	$40x^6-60x^4+21x^2$
1		x	x	$6x^3-5x$	$6x^3-5x$	$\frac{1}{3}(80x^5-112x^3+35x)$	$\frac{1}{3}(80x^5-112x^3+35x)$
0	1	1	$4x^2-3$	$4x^2-3$	$16x^4-20x^2+5$	$16x^4-20x^2+5$	$64x^6-112x^4+56x^2-7$
	0	1	2	3	4	5	6

Table of $\overleftarrow{U}_k^n(x)$

6							$2x^6$
5						$2x^5$	$2x^5$
4					$2x^4$	$2x^4$	$\frac{4}{3}(14x^6-11x^4)$
3				$2x^3$	$2x^3$	$16x^5-12x^3$	$16x^5-12x^3$
2			$2x^2$	$2x^2$	$\frac{4}{3}(10x^4-7x^2)$	$\frac{4}{3}(10x^4-7x^2)$	$\frac{1}{5}(336x^6-216x^4+63x^2)$
1		$2x$	$2x$	$\frac{4}{3}(8x^3-5x)$	$\frac{4}{3}(8x^3-5x)$	$48x^5-56x^3+14x$	$48x^5-56x^3+14x$
0	2	2	$8x^2-4$	$8x^2-4$	$32x^4-32x^2+6$	$32x^4-32x^2+6$	$128x^6-192x^4+80x^2-8$
	0	1	2	3	4	5	6

Determination of the reverse orthogonal polynomials

It is well-known that orthogonal polynomials satisfy a three-term recurrence relation. We have a similar result for reverse orthogonal polynomials, which plays a key role.

Theorem 3. *For any integer $n \geq 2$ and any integer k with $0 \leq k \leq n-2$, there are unique real numbers α_k^n and β_k^n such that*

$$\overleftarrow{F}_k^n(x) = \alpha_k^n \overleftarrow{F}_k^{n-2}(x) + \beta_k^n x \overleftarrow{F}_{k+1}^{n-1}(x).$$

From Theorem 1(c) we see that it suffices to consider the case $k \equiv n \pmod 2$, so we make that restriction.

We note that the low-order terms of $\overleftarrow{F}_k^n(x)$ and $\overleftarrow{F}_k^{n-2}(x)$ are of degree k while the low-order term of $x\overleftarrow{F}_{k+1}^{n-1}(x)$ is of degree $k+2$, and the high-order terms of $\overleftarrow{F}_k^n(x)$ and $x\overleftarrow{F}_{k+1}^{n-1}(x)$ are of degree n while the high-order term of $\overleftarrow{F}_k^{n-2}(x)$ is of degree $n-2$. Hence we must have

$$\alpha_k^n = \frac{\text{trailing coefficient of } \overleftarrow{F}_k^n(x)}{\text{trailing coefficient of } \overleftarrow{F}_k^{n-2}(x)},$$

$$\beta_k^n = \frac{\text{leading coefficient of } \overleftarrow{F}_k^n(x)}{\text{leading coefficient of } \overleftarrow{F}_{k+1}^{n-1}(x)}.$$

We also observe that each of the following sets determines the others:

(a) $\{\overleftarrow{F}_k^n(x)\}$ for all n,k.

(b1) $\{\alpha_k^n\}$ and $\{\beta_k^n\}$ for all n,k, and $\{\overleftarrow{F}_n^n(x)\}$ for all n.

(b2) $\{\alpha_k^n\}$ and $\{\beta_k^n\}$ for all n,k, and $\{\overleftarrow{F}_0^n(x)\}$ for all n.

We do some careful but routine computations of the reverse orthogonal polynomials via the Gram-Schmidt process. We obtain:

Reverse Legendre polynomials:
(a) $\overleftarrow{P}^n_n(x) = x^n$.
(b) $\overleftarrow{P}^n_{n-2}(x) = \frac{1}{2}((2n+1)x^n - (2n-1)x^{n-2})$.
(c) $\overleftarrow{P}^n_{n-4}(x) = \frac{1}{8}((2n-1)(2n+1)x^n - 2(2n-3)(2n-1)x^{n-2} + (2n-5)(2n-3)x^{n-4})$.
(d) $\overleftarrow{P}^n_{n-6}(x) = \frac{1}{48}((2n-3)(2n-1)(2n+1)x^n - 3(2n-5)(2n-3)(2n-1)x^{n-2}$
$+ 3(2n-7)(2n-5)(2n-3)x^{n-4} - (2n-9)(2n-7)(2n-5)x^{n-6})$.

Reverse Chebyshev polynomials of the first kind:
(a) $\overleftarrow{T}^n_n(x) = x^n$.
(b) $\overleftarrow{T}^n_{n-2}(x) = 2nx^n - (2n-1)x^{n-2}$.
(c) $\overleftarrow{T}^n_{n-4}(x) = \frac{1}{3}(4(n-1)nx^n - 4(n-1)(2n-3)x^{n-2} + (2n-5)(2n-3)x^{n-4})$.
(d) $\overleftarrow{T}^n_{n-6}(x) = \frac{1}{15}(8(n-2)(n-1)nx^n - 12(n-2)(n-1)(2n-5)x^{n-2}$
$+ 6(n-2)(2n-7)(2n-5)x^{n-4} - (2n-9)(2n-7)(2n-5)x^{n-6})$.

Reverse Chebyshev polynomials of the second kind:
(a) $\overleftarrow{U}^n_n(x) = 2x^n$.
(b) $\overleftarrow{U}^n_{n-2}(x) = \frac{4}{3}(2(n+1)x^n - (2n-1)x^{n-2})$.
(c) $\overleftarrow{U}^n_{n-4}(x) = \frac{2}{5}(4n(n+1)x^n - 4n(2n-3)x^{n-2} + (2n-5)(2n-3)x^{n-4})$.
(d) $\overleftarrow{U}^n_{n-6}(x) = \frac{8}{105}(8(n-1)n(n+1)x^n - 12(n-1)n(2n-5)x^{n-2}$
$+ 6(n-1)(2n-7)(2n-5)x^{n-4} - (2n-9)(2n-7)(2n-5)x^{n-6})$.

Lemma 4. *(a) For any integer $n \geq 2$ and any integer k with $0 \leq k \leq n-2$, $k \equiv n \pmod 2$*

$$\overleftarrow{P}_k^n(x) = \alpha_k^n \overleftarrow{P}_k^{n-2}(x) + \beta_k^n x \overleftarrow{P}_{k+1}^{n-1}(x).$$

for

$$\alpha_k^n = -\frac{n+k+1}{n-k} \quad \textit{and} \quad \beta_k^n = \frac{2n+1}{n-k}.$$

(b) For any integer $n \geq 2$ and any integer k with $0 \leq k \leq n-2$, $k \equiv n \pmod 2$,

$$\overleftarrow{T}_k^n(x) = \alpha_k^n \overleftarrow{T}_k^{n-2}(x) + \beta_k^n x \overleftarrow{T}_{k+1}^{n-1}(x).$$

for

$$\alpha_k^n = -\frac{n+k+1}{n-k-1} \quad \textit{and} \quad \beta_k^n = \frac{2n}{n-k-1}.$$

(c) For any integer $n \geq 2$ and any integer k with $0 \leq k \leq n-2$, $k \equiv n \pmod 2$

$$\overleftarrow{U}_k^n(x) = \alpha_k^n \overleftarrow{U}_k^{n-2}(x) + \beta_k^n x \overleftarrow{U}_{k+1}^{n-1}(x).$$

for

$$\alpha_k^n = -\frac{(n-k+2)(n+k+1)}{(n-k)(n-k+1)} \quad \textit{and} \quad \beta_k^n = \frac{2(n-k+2)(n+1)}{(n-k)(n-k+1)}.$$

For an integer t, we denote the product of m consecutive integers, the largest of which is t, by $\Gamma_m(t)$, and for an odd integer t, we denote the product of m consecutive odd integers, the largest of which is t, by $\Pi_m(t)$, i.e.,

$$\Gamma_0(t) = 1 \qquad \text{and} \qquad \Gamma_m(t) = \prod_{r=1}^{m} (t-(r-1)) \quad \text{for} \quad m > 0,$$

$$\Pi_0(t) = 1 \qquad \text{and} \qquad \Pi_m(t) = \prod_{r=1}^{m} (t-2(r-1)) \quad \text{for} \quad m > 0.$$

Theorem 5. *(a) For any integer $n \geq 0$ and any integer k with $0 \leq k \leq n$, $k \equiv n \pmod 2$, let $m = (n-k)/2$. Then the reverse Legendre polynomial $\overleftarrow{P}_k^n(x)$ is given by*

$$\overleftarrow{P}_k^n(x) = \frac{1}{2^m}\frac{1}{m!}\left(\sum_{p=0}^{m}(-1)^p\binom{m}{p}\Pi_m(2n-2p+1)x^{n-2p}\right)$$

(b) For any integer $n \geq 0$ and any integer k with $0 \leq k \leq n$, $k \equiv n \pmod 2$, let $m = (n-k)/2$. Then the reverse Chebyshev polynomial of the first kind $\overleftarrow{T}_k^n(x)$ is given by

$$\overleftarrow{T}_k^n(x) = \frac{1}{\Pi_m(2m-1)}\left(\sum_{p=0}^{m}(-1)^p 2^{m-p}\binom{m}{p}\Gamma_{m-p}(n-p)\Pi_p(2n-2m+1)x^{n-2p}\right)$$

(c) For any integer $n \geq 2$ and any integer k with $0 \leq k \leq n$, $k \equiv n \pmod 2$, let $m = (n-k)/2$. Then the reverse Chebyshev polynomial of the second kind $\overleftarrow{U}_k^n(x)$ is given by

$$\overleftarrow{U}_k^n(x) = \frac{2(m+1)}{\Pi_{m+1}(2m+1)}\left(\sum_{p=0}^{m}(-1)^p 2^{m-p}\binom{m}{p}\Gamma_{m-p}(n+1-p)\Pi_p(2n-2m+1)x^{n-2p}\right)$$

Remark 6. The coefficients of $\overleftarrow{P}_k^n(x)$, $\overleftarrow{T}_k^n(x)$, and $\overleftarrow{U}_k^n(x)$ are rational numbers.

We conjectured that the coefficients of $\overleftarrow{P}_k^n(x)$ have denominators a power of 2 and this was originally proved by T. Amdeberhan and V. Moll. But we can easily see that this is true as it is an elementary fact that for any integer $m \geq 0$ and any integer t, $2^m\Pi_m(t)$ is divisible by $m!$.

On the other hand, the coefficients of $\overleftarrow{T}_k^n(x)$ and $\overleftarrow{U}_k^n(x)$ are not always integers but always have denominators odd integers.