The Geometry of the Hyperelliptic Torelli Group

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Study of Surfaces

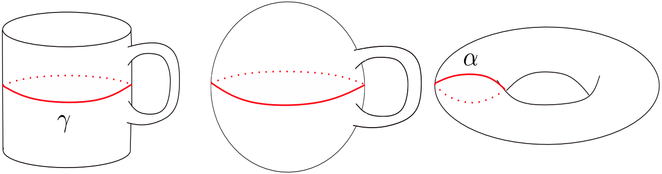
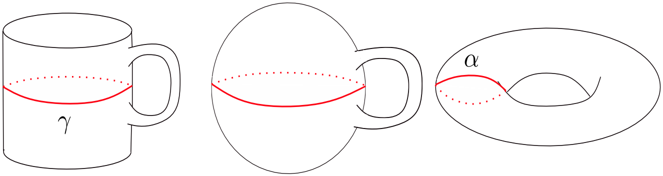
- 

Figure: A deformation of a Coffee Mug

Study of Surfaces

- 

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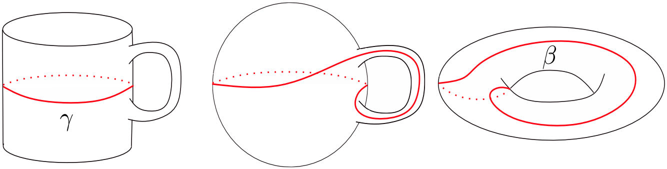
- 

Figure: Another deformation of a Coffee Mug

A Reference Surface

Let $S^r_{g,n}$ be an oriented topological surface with g holes, n punctures, and r boundary components:

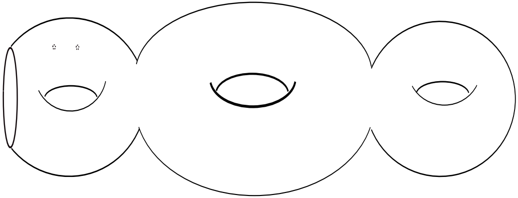


Figure: $S^1_{3,2}$

A Diffeomorphism of a Surface with Two Holes

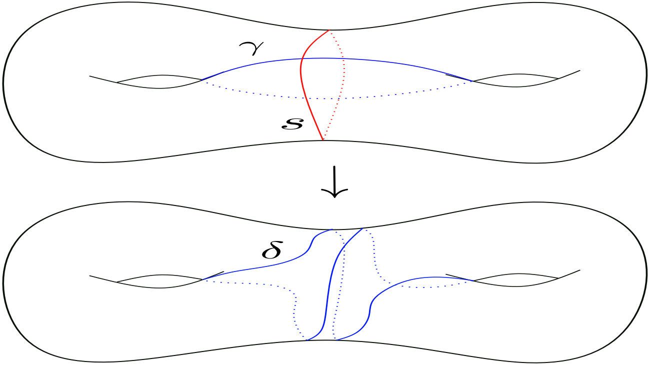


Figure: Twisting of a surface S_2

Monodromy and Deformations

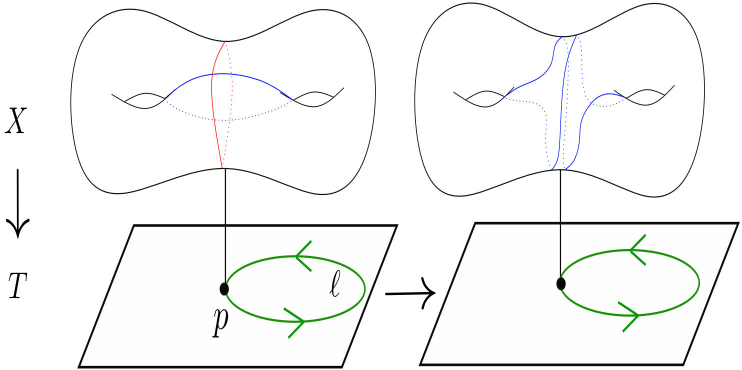


Figure: The monodromy action

Complex Curves

A complex curve C is an algebraic curve defined over the complex numbers $\mathbb{C}$:

$$C = \text{ the zero set of polynomial equations over } \mathbb{C} \text{ of dim } 1$$

A smooth connected complex curve C is diffeomorphic to $S_{g,n}$ for some g and n.

- In $\mathbb{C}^2$, $C = \{f(x,y) = 0\}$, where $f(x,y) \in \mathbb{C}[x,y]$.
- The integer g is called the genus of C.
- Filling the n punctures makes C compact, and in fact, projective.

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The Fundamental Group of a Top Surface

For a path-connected topological space X, the fundamental group of X is defined to be

$$\pi_1(X) := \text{ homotopy classes of loops in } X$$

Its abelianization is isomorphic to $H_1(X)$:

$$\pi_1^{\mathrm{ab}}(X) \cong H_1(X).$$

$$\pi_1(S_g) \cong \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g | \prod_{j=1}^{g} [\alpha_j, \beta_j] = e \rangle$$

$$\pi_1^{\mathrm{ab}}(S_g) \cong H_1(S_g) = \mathbb{Z}^{2g}$$

$H_1(S_g)$ with the intersection pairing $\langle \ , \ \rangle$ = a symplectic space of rank $2g$

Loop Multiplication

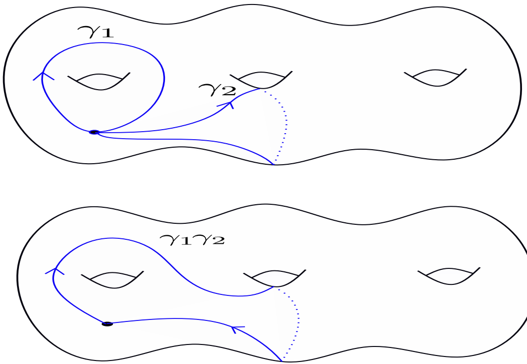

Figure: Loop Multiplication for $\pi_1(X)$

Standard Generators for $\pi_1(S_g)$

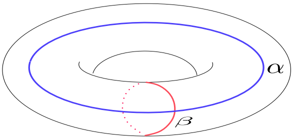

Figure: A torus with generators for $\pi_1(\mathbb{T})$

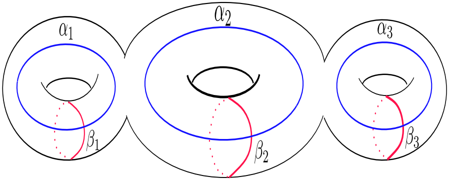

Figure: A surface of genus 3 with generators for $\pi_1(S_3)$

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Definition

The mapping class group $\Gamma^r_{g,n}$ of $S^r_{g,n}$ is defined to be

$$\Gamma^r_{g,n} := \mathrm{Diff}^+(S^r_{g,n})/\cong,$$

where $\cong$ is the isotopy relation.

Notes:

- The punctures and boundary components are fixed pointwise.
- Forgetting the n punctures and r boundary components gives the projection

$$\mathcal{F}\mathrm{orget} : \Gamma^r_{g,n} \to \Gamma_g$$

Dehn Twists

Definition

A Dehn twist T_d about a simple closed curve d is an isotopy class of a left-twist map about γ, fixing the boundary of a tubular neighborhood N of d.

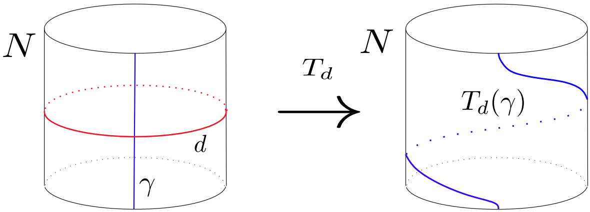

Figure: A View of a Dehn Twist

Symplectic Representation of $\Gamma_{g,n}$

Fix a symplectic basis $a_1, b_1 \ldots, a_g, b_g$ for $H_1(S_g)$
Here $a_i = [\alpha_i]$ and $b_i = [\beta_i]$.

$$\mathrm{Aut}(H_1(S_g), \langle\ ,\ \rangle) \cong \mathrm{Sp}_{2g}(\mathbb{Z})$$

$\mathrm{Sp}_{2g}(\mathbb{Z})$ = the group of $2g$-by-$2g$ symplectic matrices with integer entries

The mapping class group $\Gamma_{g,n}$ acts on $\pi_1(S_g)$ and so $H_1(S_g)$ preserving $\langle\ ,\ \rangle$:

$$\rho : \Gamma_{g,n} \to \mathrm{Sp}_{2g}(\mathbb{Z}),$$

Theorem

The representation ρ is surjective for $g \geq 1$.

The Push Map

There is a natural inclusion map

$$\mathcal{P}\mathrm{ush} : \pi_1(S_g) \to \Gamma_{g,1}.$$

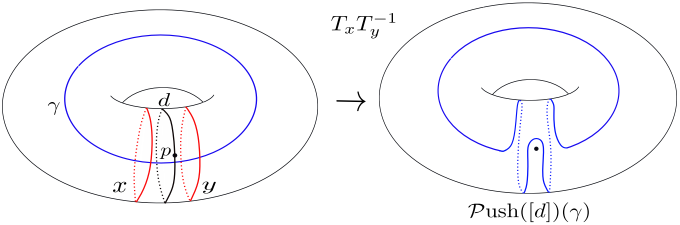

Figure: The point-pushing map $\mathcal{P}\mathrm{ush}$

The Birman Exact Sequence

Composing the Push map with the Forget map, we obtain the Birman exact sequence

$$1 \to \pi_1(S_g) \to \Gamma_{g,1} \to \Gamma_g \to 1.$$

This extends to the punctured surface $S_{g,n}$:

$$1 \to \pi_1(S_{g,n}) \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1.$$

Theorem

If $g \geq 4$ and $n \geq 0$, the Birman exact sequence does not split.

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The Kernel of ρ

Definition

The Torelli group $T_{g,n}$ is defined to be

$$T_{g,n} := \ker(\rho : \Gamma_{g,n} \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}))$$

There is the short exact sequence:

$$1 \to T_g \to \Gamma_g \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1.$$

Theorem (Johnson, Putman)

- For $g \geq 3$, T_g is generated by a finite number of bounding pair maps and separating twists.

Generators of T_g

- T_s is called a separating twist when a simple closed curve s disconnects the surface.
- $T_a T_b^{-1}$ is called a bounding pair map when a and b are homologus.

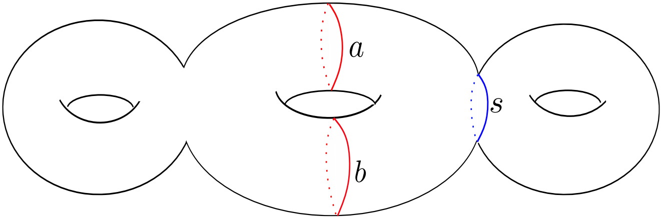

Figure: A bounding pair and a separating curve

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A Hyperelliptic Involution

A hyperelliptic involution $\sigma : S_g \to S_g$ is an orientation preserving diffeomorphism of order 2 of S_g fixing exactly $2g+2$ points.

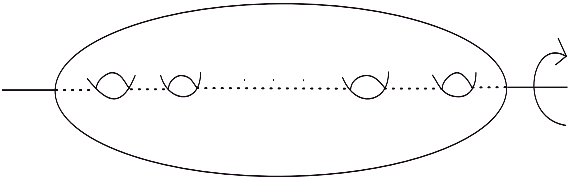

Figure: A hyperelliptic involution of S_g, rotation by π

Fix a hyperelliptic involution σ of S_g.

Definition

The hyperelliptic mapping class group Δ_g of S_g is defined to be

$$\Delta_g := \text{the centralizer of the isotopy class of } \sigma \text{ in } \Gamma_g$$

Facts:

- $\Delta_2 = \Gamma_2$
- Δ_g is an infinite-index subgroup of Γ_g

Generators of Δ_2

Fact: For any $f \in \Gamma_g$,

$$f \text{ commutes with } T_a \iff f(a) = a$$

A simple closed curve γ is said to be symmetric if $\sigma(\gamma) = \gamma$.

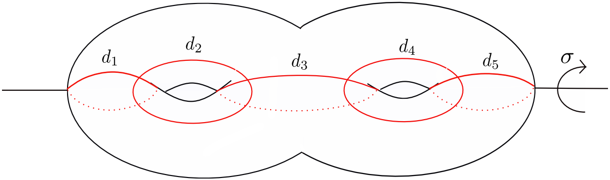

Figure: Nonseparating symmetric curves generating Δ_2

Generators of Δ_g

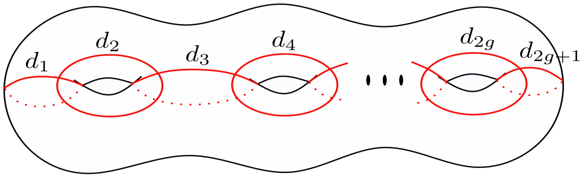

Figure: Nonseparating symmetric curves generating generating Δ_g

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The Hyperelliptic Torelli Group $T\Delta_g$

Definition

The hyperelliptic Torelli group $T\Delta_g$ is defined to be

$$T\Delta_g := T_g \cap \Delta_g.$$

- $T\Delta_g$ is an infinite-index subgroup of T_g.

Theorem (Brendle-Margalit-Putman)

If $g \geq 2$, then $T\Delta_g$ is generated by Dehn twists about symmetric separating curves.

Symmetric Separating Curves

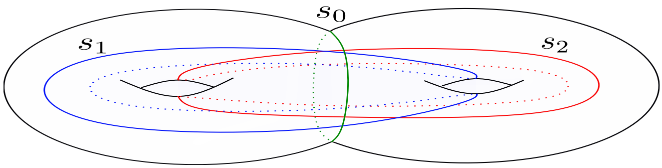

Figure: Symmetric separating curves in S_2

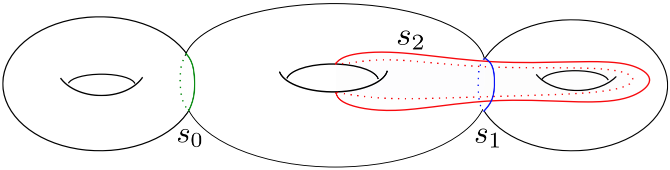

Figure: Symmetric separating curves in S_3

Open Problems

1. Is $T\Delta_g$ finitely generated for $g \geq 3$?

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2. Is $T\Delta_g$ finitely presented for $g \geq 4$?

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Open Problems

1. Is $T\Delta_g$ finitely generated for $g \geq 3$?
2. Is $T\Delta_g$ finitely presented for $g \geq 4$?
3. Determine $H_1(T\Delta_g; \mathbb{Z})$ and $H_1(T\Delta_g; \mathbb{Q})$.

Remarks:

1. $T\Delta_2 = T_2$ is an infinitely generated free group (Mess).
2. $T\Delta_3$ is not finitely presentable(Brendle-Childers-Margalit).
3. The study of the abelianization involves a certain algebraic cycle and is useful in the study of the moduli stack of hyperelliptic curves.