A Low-cost Algorithm to Determine Orbital Trajectories within the Cislunar Region

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Exchange of Mathematical Ideas
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Outline

- Background
- Orbits in the CR3BP
- Spacecraft Trajectories
- Algorithm for Spacecraft Trajectories
- Numerical Simulations

Needless to Determine Spacecraft Trajectories

- **The Cislunar region** is gaining increased attention throughout the past few years, **as 90 missions to the Moon are projected by 2030** with additional missions to Mars.
- As **traffic in the Cislunar region** continues to grow, **efficient methods to propagate trajectories**, win the circular restricted three-body problem (CR3BP) are required.
- Many upcoming Cislunar missions are focused on the Lunar South pole, as well as **periodic orbits about L$_1$ and L$_2$** of the **Earth-Moon CR3BP** system.
- Russia's Luna 25 (2022), South Korea's KPLO (2022), Japan and India's joint LUPEX (2023), and India's Chandrayaan-3 (2024) all are **missions** to **observe or land on polar regions of the Moon**.
- **NASA's Artemis program** is a **multi-stage program** to reestablish presence on the **Moon**.
- **The dynamics** that govern the motion in the **CR3BP are highly non-linear and no closed form solution** has yet been derived.
- To be able to **design trajectories** in such a model, **different numerical methods:** Gauss-Legendre, Dormand-Prince, and Chebyshev-Picard, Gragg-Bulirsch-Stoer, Adams-Bashforth, or Runge-Kutta integrator, are required to plan space trajectories that satisfy desired behaviors.

A solution is proposed for the well-known CR3BP to determine trajectories via a low-complexity algorithm.

The Cislunar Region

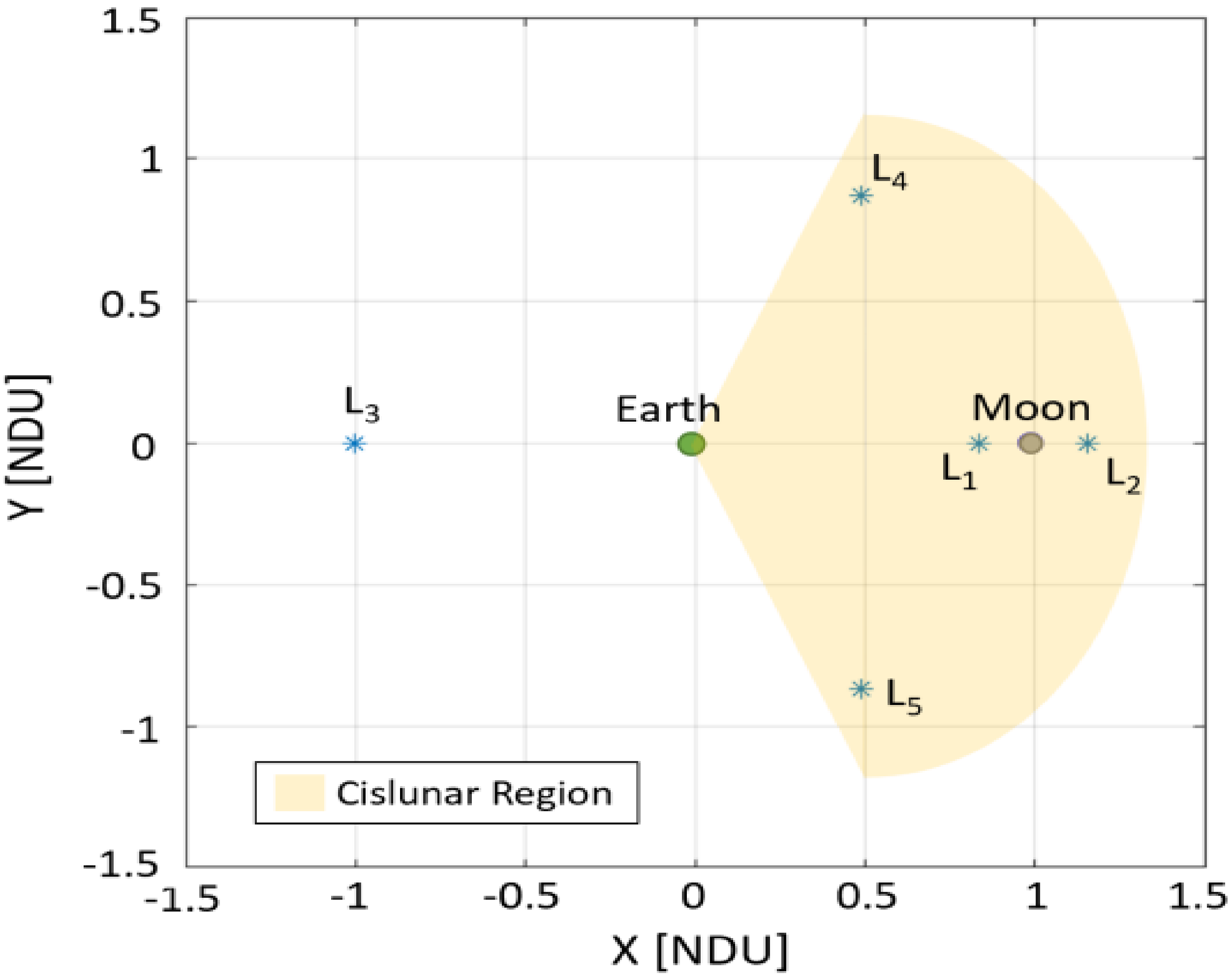


Fig. 1: Cislunar Region.

Exploring Structures

- To produce **computationally tractable and inexpensive algorithm**, particularly in the **complex and chaotic three-body dynamics**, it is important to address system structures in relevant equations, development of **novel theories**, and design **low-complexity and reliable algorithm**.
- Many problems in applied sciences and engineering can be reduced to **linear algebra problem**.
- Standard methods may not be practical due to **large dense matrices**.
- **Exploiting the structure lead what???**
- **Speed**

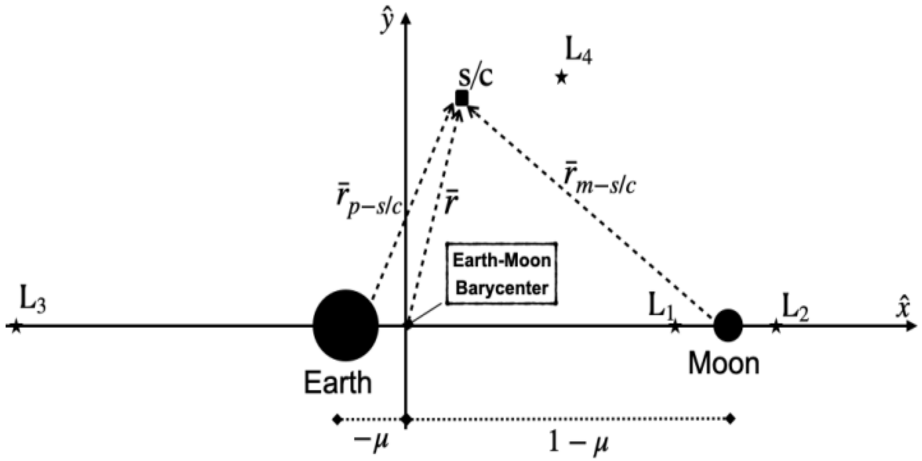
Standard algorithms	$\mathcal{O}(n^3)$ arithmetic operations
Fast algorithms	$\mathcal{O}(n^2)$
Faster algorithms	$\mathcal{O}(n\,log(n))$
Superfast algorithms	$\mathcal{O}(nlog(log(n)))$

Obtain Periodic Orbits in the CR3BP

- The **evolution of a spacecraft (s/c) position** $\bar{r}_{rot} = [x, y, z]^T$ and **velocity** $\dot{\bar{r}}_{rot} = [\dot{x}, \dot{y}, \dot{z}]^T$ is governed by the following equations of motion:

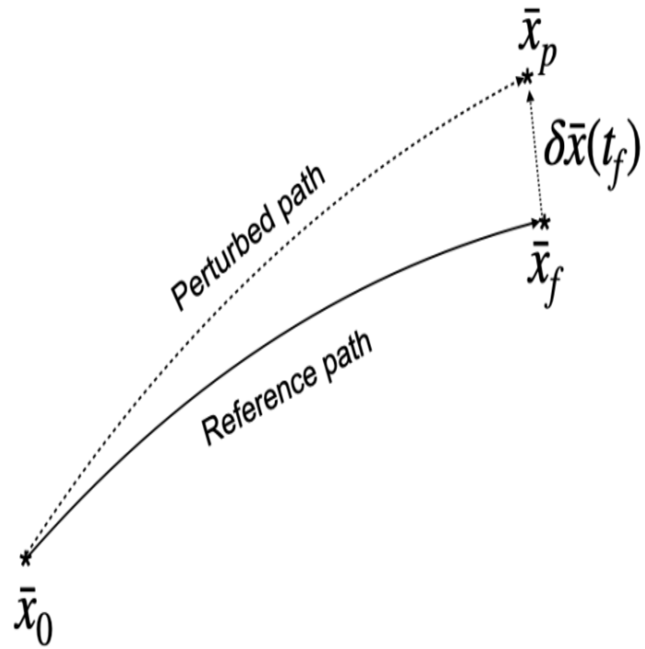
$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}; \; \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}; \; \ddot{z} = \frac{\partial U^*}{\partial z}$$

where $U^* := \frac{1-\mu}{r_{p-s/c}} + \frac{\mu}{r_{m-s/c}} + \frac{1}{2}(x^2 + y^2)$ represents the pseudo-potential function, mass ratio $\mu = m_M/(m_M + m_E)$ is defined for the system, with m_M and m_E being the masses of the Moon and the Earth, and $r_{p-s/c}$ and $r_{m-s/c}$ are the distances of the s/c to the Earth and the Moon, respectively.



Obtain Periodic Orbits in the CR3BP

- **Spacecraft trajectory design** is mostly based on **numerical strategies: differential corrections** to find trajectories that satisfy specific purposes in different models.
- It is important to find a **correlation** between the variations in the **initial state of a trajectory**, $\delta\bar{x}_0$, with the variations of its **final states**, $\delta\bar{x}_f$, where $\delta\bar{x}(t) = [\delta x, \delta y, \delta z, \delta\dot{x}, \delta\dot{y}, \delta\dot{z}]^T$



- **State Transition Matrix (STM)** is a variable sensitivity matrix and very useful for targeting schemes and stability analysis: $\phi(t_f, t_0) = \frac{\partial\bar{x}(t_f)}{\partial\bar{x}(t_0)}$

Obtain Periodic Orbits in the CR3BP

- The **linear variational equations**, derived from the equations of motion, are provided in the form:

$$\delta\dot{\bar{x}}(t) = \mathbf{A}(t)\delta\bar{x}(t),$$

where $\mathbf{A}(t)$ is the Jacobian matrix comprised of the partials of the equations of motion with respect to the states evaluated at the time t: $\mathbf{A}(t) = \frac{\partial \bar{f}(\bar{x},t)}{\partial \bar{x}(t)}$ and $\dot{\bar{x}} = \bar{f}(\bar{x},t)$.

- The evolution of $\phi(t,t_0)$ is governed by the following matrix differential equation:

$$\dot{\phi}(t,t_0) = \mathbf{A}(t)\phi(t,t_0)$$

- In the CR3BP,

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \\ \delta\ddot{x} \\ \delta\ddot{y} \\ \delta\ddot{z} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ U^*_{xx} & U^*_{xy} & U^*_{xz} & 0 & 2 & 0 \\ U^*_{xy} & U^*_{yy} & U^*_{yz} & -2 & 0 & 0 \\ U^*_{xz} & U^*_{yz} & U^*_{zz} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix}$$

Periodic Orbits via a Multi-variable Newton-Raphson

- Consider that $\bar{\mathcal{Y}}$ contains free variables and the constraints $\bar{\mathcal{F}}(\bar{\mathcal{Y}})$ are defined via $\bar{\mathcal{Y}} = \begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \vdots \\ \mathcal{Y}_n \end{pmatrix}$ and $\bar{\mathcal{F}}(\bar{\mathcal{Y}}) = \begin{pmatrix} \mathcal{F}_1(\bar{\mathcal{Y}}) \\ \mathcal{F}_2(\bar{\mathcal{Y}}) \\ \vdots \\ \mathcal{F}_n(\bar{\mathcal{Y}}) \end{pmatrix}$, respectively.
- The goal is to find $\bar{\mathcal{Y}}$ that makes the vector of constraints null: $\bar{\mathcal{F}}(\bar{\mathcal{Y}}) = \bar{0}$ with accuracy: $|\bar{\mathcal{F}}| < \epsilon$ where $\epsilon = 10^{-12}$.
- Update the NR algorithm

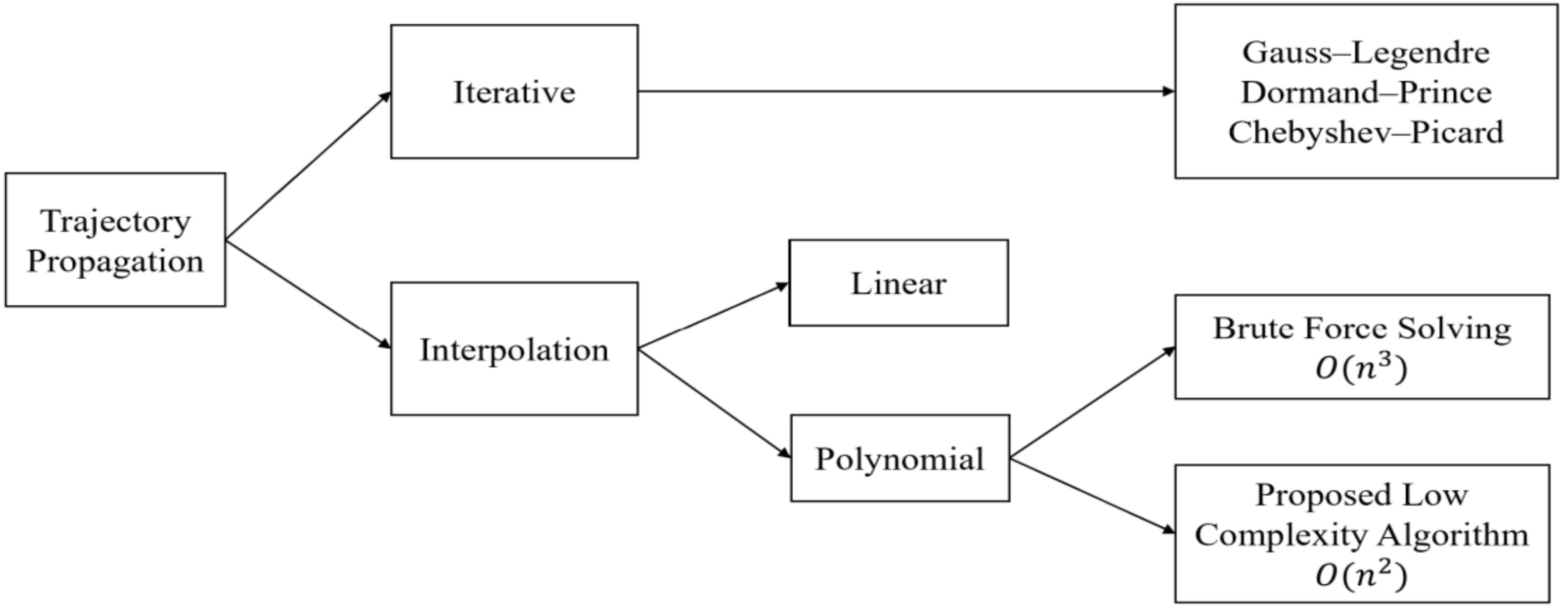
$$\bar{\mathcal{Y}} = \bar{\mathcal{Y}}_0 - D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0)^T [D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0) D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0)^T]^{-1} \bar{\mathcal{F}}(\bar{\mathcal{Y}})$$

where $n \neq j$ and

$$D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0) = \frac{\partial \bar{\mathcal{F}}(\bar{\mathcal{Y}})_0}{\partial \bar{\mathcal{Y}}_0} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial \mathcal{Y}_1} & \frac{\partial \mathcal{F}_1}{\partial \mathcal{Y}_2} & \cdots & \frac{\partial \mathcal{F}_1}{\partial \mathcal{Y}_n} \\ \frac{\partial \mathcal{F}_2}{\partial \mathcal{Y}_1} & \frac{\partial \mathcal{F}_2}{\partial \mathcal{Y}_2} & \cdots & \frac{\partial \mathcal{F}_2}{\partial \mathcal{Y}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{F}_j}{\partial \mathcal{Y}_1} & \frac{\partial \mathcal{F}_j}{\partial \mathcal{Y}_2} & \cdots & \frac{\partial \mathcal{F}_j}{\partial \mathcal{Y}_n} \end{bmatrix}_0 .$$

Comparison between Iterative and Proposed Interpolation Trajectory Propagation Methods

Interpolation	Iterative
Continuous	Discrete
Best-fit Solution	Approximated Solution
Low-complexity	Computationally expensive



Proposed Spacecraft Trajectories

- Assume that the positions, velocities, and accelerations of each spacecraft at n time intervals are known, and their magnitudes are given via $(t_i^{(j)}, \mid \ \bar{x}_j(t_i^{(j)}) \ \mid), (t_i^{(j)}, \mid \dot{\bar{x}}_j(t_i^{(j)}) \mid)$, and $(t_i^{(j)}, \mid \ddot{\bar{x}}_j(t_i^{(j)}) \mid)$ for $i = 0, 1, \cdots, n$ and $t_0^{(j)} < t_1^{(j)} < \cdots < t_n^{(j)}$, where $j = 1, 2, \cdots, m$, and $\bar{x}_j(t_i^{(j)}), \dot{\bar{x}}_j(t_i^{(j)})$, and $\ddot{\bar{x}}_j(t_i^{(j)})$ are vectors in $\mathbb{R}^3$ evaluated at each $t_i^{(j)}$.
- The equations of motion or **trajectories of m spacecrafts** over n time intervals are described via piecewise-defined functions on the interval $[t_k^{(j)}, t_{k+1}^{(j)}]$ from $\mathbb{R}^3$ to $\mathbb{R}$ such that

$$G_k(|\bar{x}_j(t^{(j)})|) = g_{0,k} + g_{1,k}t^{(j)} + g_{2,k}(t^{(j)})^2 + g_{3,k}(t^{(j)})^3 + g_{4,k}(t^{(j)})^4 + g_{5,k}(t^{(j)})^5,$$

where $t_k^{(j)} \leqslant t^{(j)} \leqslant t_{k+1}^{(j)}$, $k = 0, 1, \cdots, n-1$, and $g_{0,k}, g_{1,k}, \cdots, g_{5,k}$ are constants that depend on the magnitude of $\bar{x}_j(t^{(j)}), \dot{\bar{x}}_j(t^{(j)})$, and $\ddot{\bar{x}}_j(t^{(j)})$.

Propose Spacecraft Trajectories

At the time interval $[t_k^{(j)}, t_{k+1}^{(j)}]$

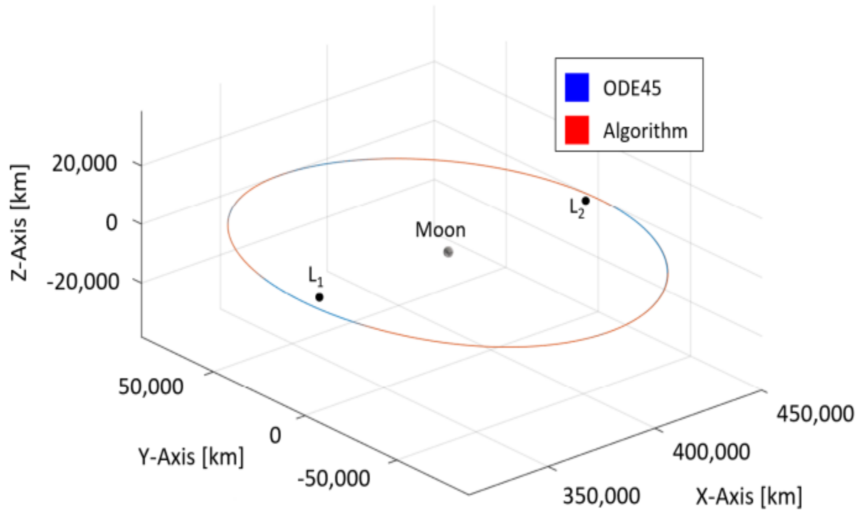
$$\underbrace{\begin{bmatrix} 1 & t_k^{(j)} & (t_k^{(j)})^2 & (t_k^{(j)})^3 & (t_k^{(1)})^4 & (t_k^{(j)})^5 \\ 1 & t_{k+1}^{(j)} & (t_{k+1}^{(j)})^2 & (t_{k+1}^{(j)})^3 & (t_{k+1}^{(j)})^4 & (t_{k+1}^{(1)})^5 \\ 0 & 1 & 2t_k^{(j)} & 3(t_k^{(j)})^2 & 4(t_k^{(j)})^3 & 5(t_k^{(j)})^4 \\ 0 & 1 & 2t_{k+1}^{(j)} & 3(t_{k+1}^{(j)})^2 & 4(t_{k+1}^{(j)})^3 & 5(t_{k+1}^{(j)})^4 \\ 0 & 0 & 2 & 6t_k^{(j)} & 12(t_k^{(j)})^2 & 20(t_k^{(j)})^3 \\ 0 & 0 & 2 & 6t_{k+1}^{(j)} & 12(t_{k+1}^{(j)})^2 & 20(t_{k+1}^{(j)})^3 \end{bmatrix}}_{A_k} \underbrace{\begin{bmatrix} g_{0,k} \\ g_{1,k} \\ g_{2,k} \\ g_{3,k} \\ g_{4,k} \\ g_{5,k} \end{bmatrix}}_{\underline{x}_k} = \underbrace{\begin{bmatrix} |\bar{x}_1(t_k^{(j)})| \\ |\bar{x}_1(t_{k+1}^{(j)})| \\ |\dot{\bar{x}}_1(t_k^{(j)})| \\ |\dot{\bar{x}}_1(t_{k+1}^{(j)})| \\ |\ddot{\bar{x}}_1(t_k^{(j)})| \\ |\ddot{\bar{x}}_1(t_{k+1}^{(j)})| \end{bmatrix}}_{\underline{b}_k}$$

$$\left(\prod_{r=1}^{5} L_r\right) U_k \underline{x}_k = \underline{b}_k, \quad \text{where} \quad A_k = \left(\prod_{r=1}^{5} L_r\right) U_k$$

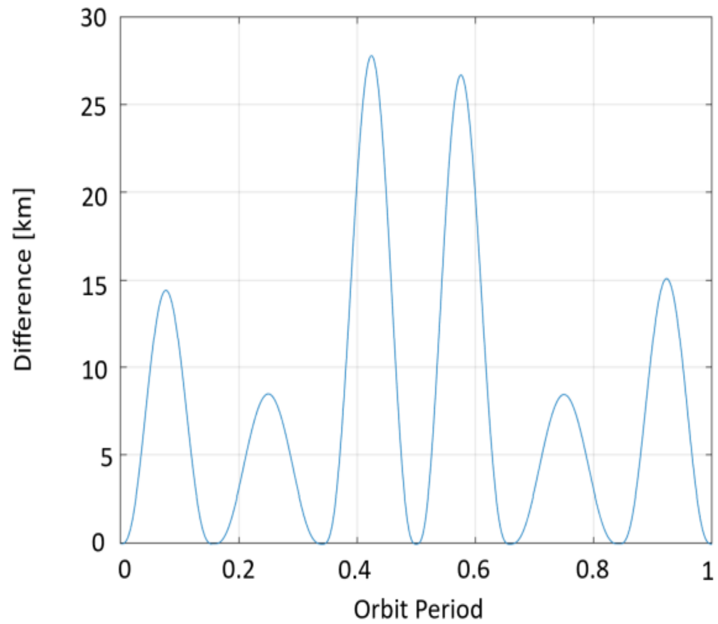
where $L_r \in \mathbb{R}^{6\times 6}$, $r = 1, 2, .., 5$ for bidiagonal lower triangular matrices, and $U_k \in \mathbb{R}^{6\times 6}$ is an upper triangular matrix.

The proposed algorithm has $\mathcal{O}(n^2)$ complexity.

Distant Retrograde Orbit



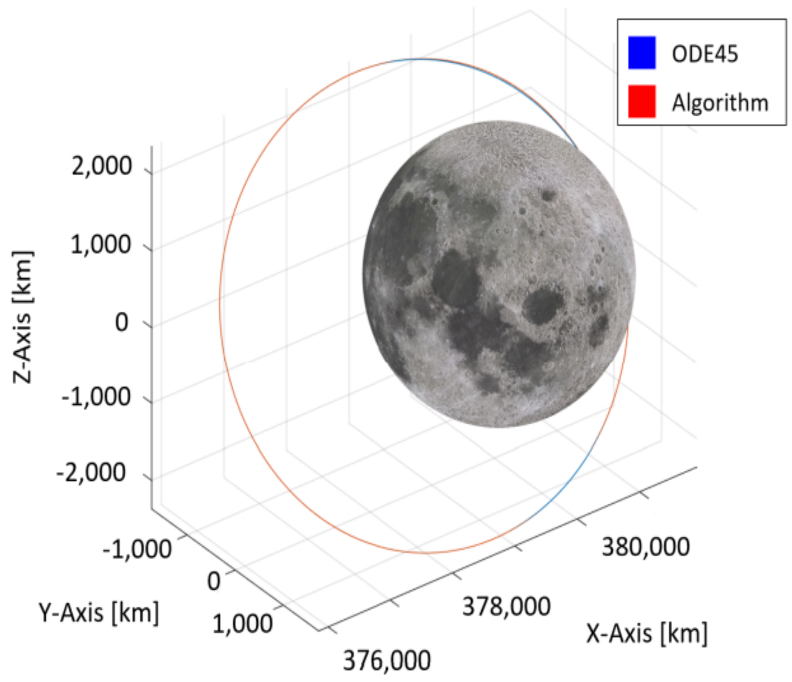
(a) Propagated DRO trajectory.



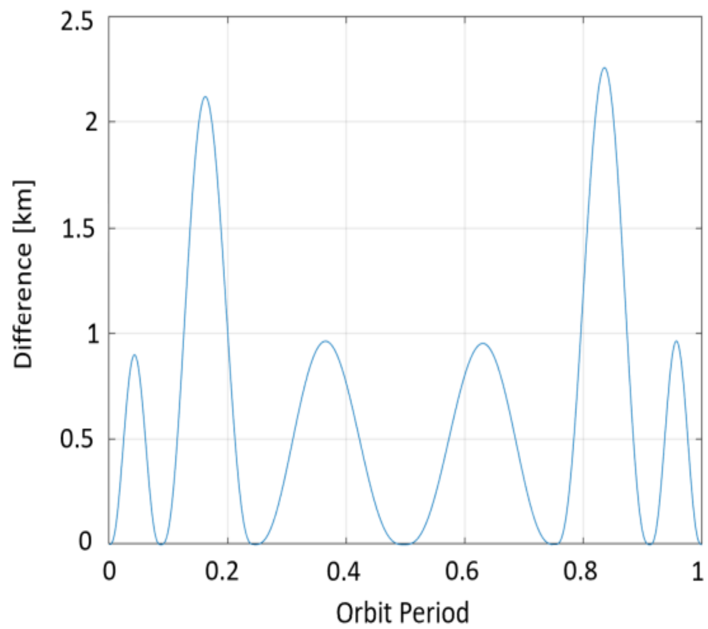
(b) Difference from LCA to ODE45.

The algorithm matches the ODE45 resolved trajectory extremely closely, deviating up to 27 kilometer. A difference of 27 kilometers between ODE45 and the LCA is insignificant in reference to the 100,000 kilometers the DRO stretches across.

Low-Lunar Orbit (LLO) Analysis



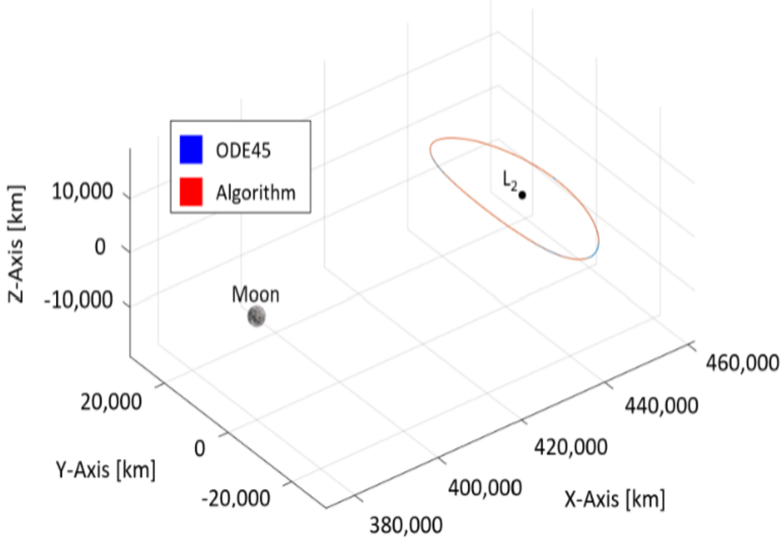
(a) Propagated LLO trajectory.



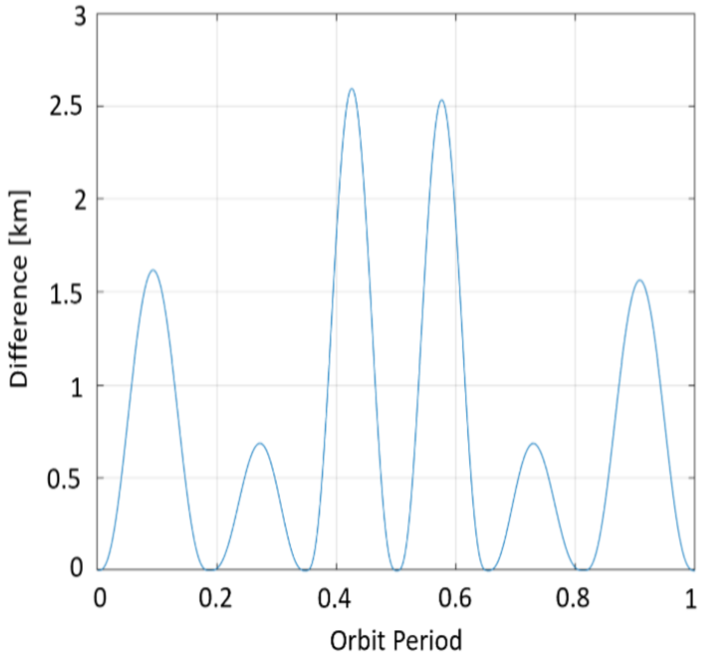
(b) Difference from LCA to ODE45.

The LLO has a peak difference of 2.3 kilometers occurring in the fifth time interval of the algorithm. The 2.3 kilometers difference is also insignificant in reference to the 4,000 kilometers the LLO stretches across.

L_2 Lyapunov Analysis



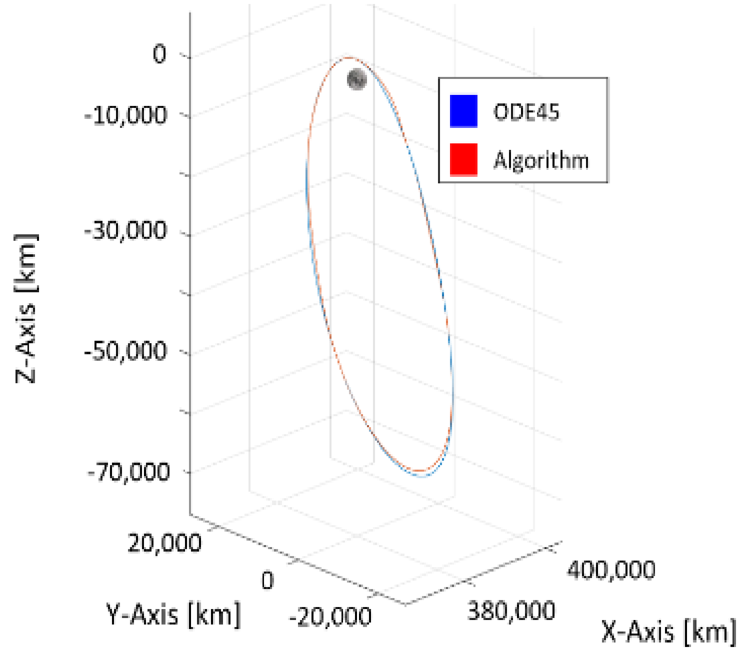
(a) Propagated Lyapunov trajectory.



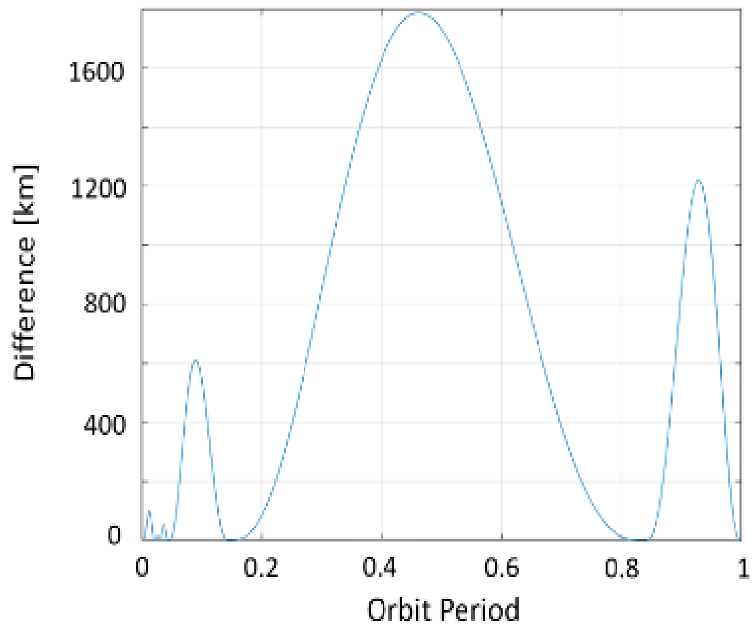
(b) Difference from LCA to ODE45.

The peak difference occurs in the third time interval set by the boundary conditions, only reaching right over 2.5 kilometers. The LCA's model of the Lyapunov orbit has the closest resemblance to the ODE45 trajectory out of any of all the tested orbits.

Near-rectilinear Halo Orbit



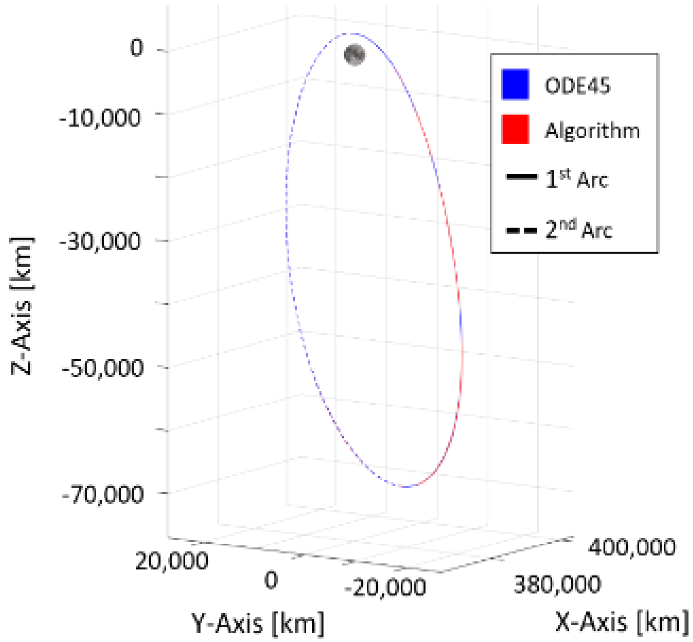
(a) Propagated NRHO trajectory.



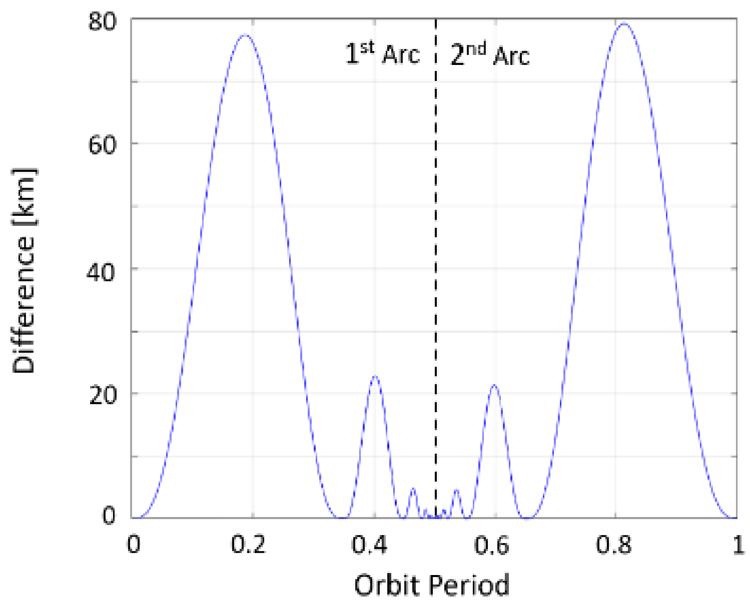
(b) Difference from LCA to ODE45.

A challenge occurred in reconstructing NRHO using the LCA.

Near-rectilinear Halo Orbit



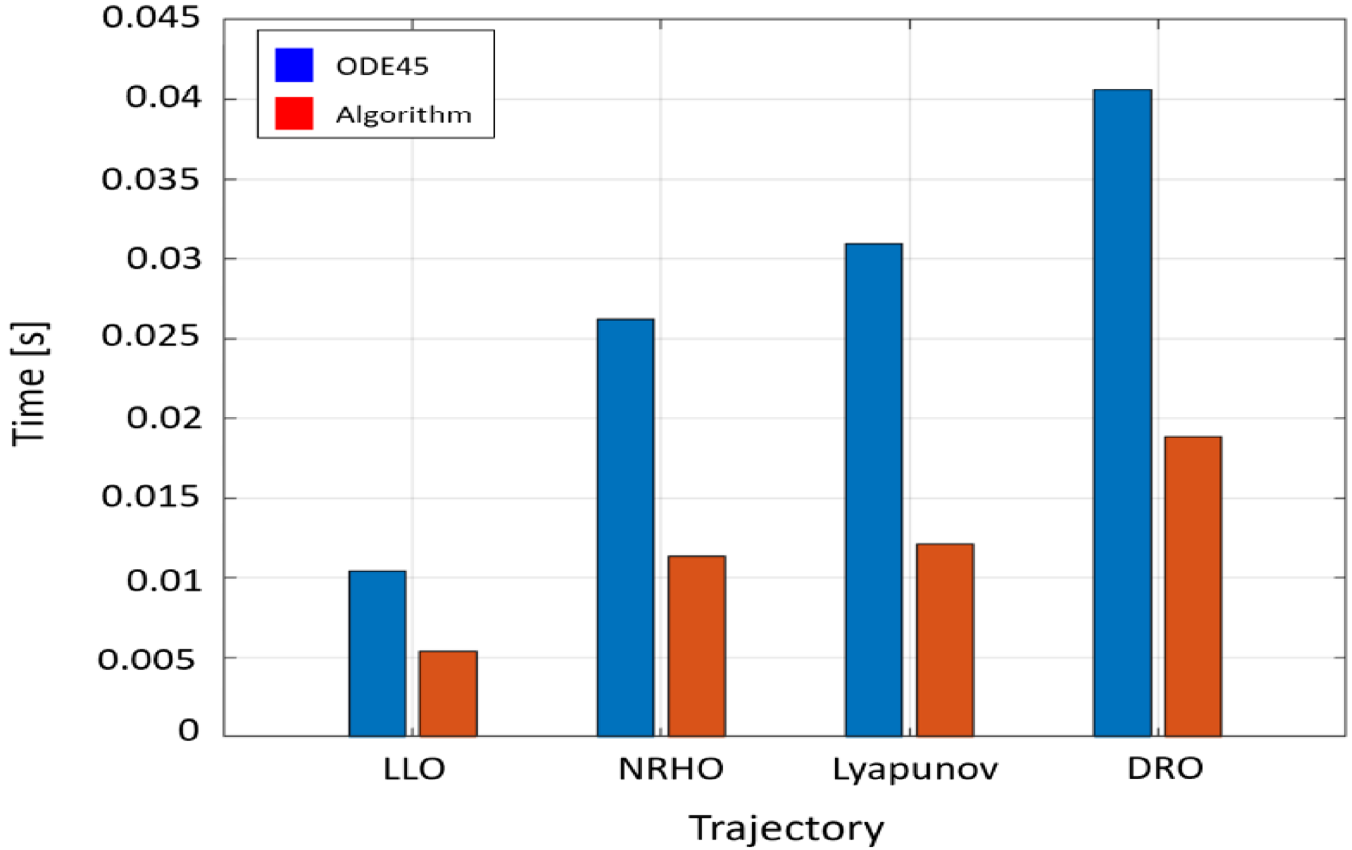
(a) Two arc NRHO trajectory.



(b) Difference from LCA to ODE45.

Breaking the trajectory into two arcs enables the maximum difference between the LCA and ODE45 propagation drops from 1800 kilometers to 78 kilometers.

Time Complexity Analysis



Average computational time of 100 propagations of each respective trajectory depicts that the algorithm resolves the trajectory significantly faster than ODE45, clocking in at about half of the time for each orbit.

A Low-cost Spacecraft Trajectories

- **Spacecraft Trajectories in the Cislunar Region**
- **A Low arithmetic-complexity Algorithm**
- **A Low time-complexity Algorithm**
- **Numerical Simulations for Accuracy**
- **Multiple Spacecraft Trajectories for Future Missions**

The First Female Recognition in the Flight Research Division, NASA

Katherine Johnson (1918 - 2020)

Johnson was an iconic woman of color in STEM. Her mathematical work helped NASA's first crewed spaceflight land on the moon in 1969.

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David Canales, Sirani M. Perera, Atahan Kurttisi, and Brian Baker-McEvilly, A Low-Complexity Algorithm to Determine Trajectories within the Circular Restricted Three-Body Problem, in the review of the Journal of the Astronautical Sciences, Springer Nature, (2023)