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Rotation Operations on the Errera Map and its Variations - Idea I

Professor Weiguo Xie, Dr. Andrew Bowling

University of Minnesota - Duluth

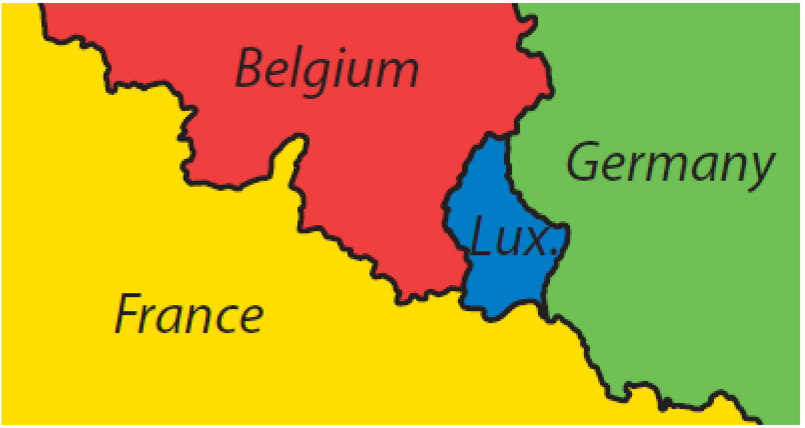
Email: xiew@umn.edu; abowling@umn.edu

Outline

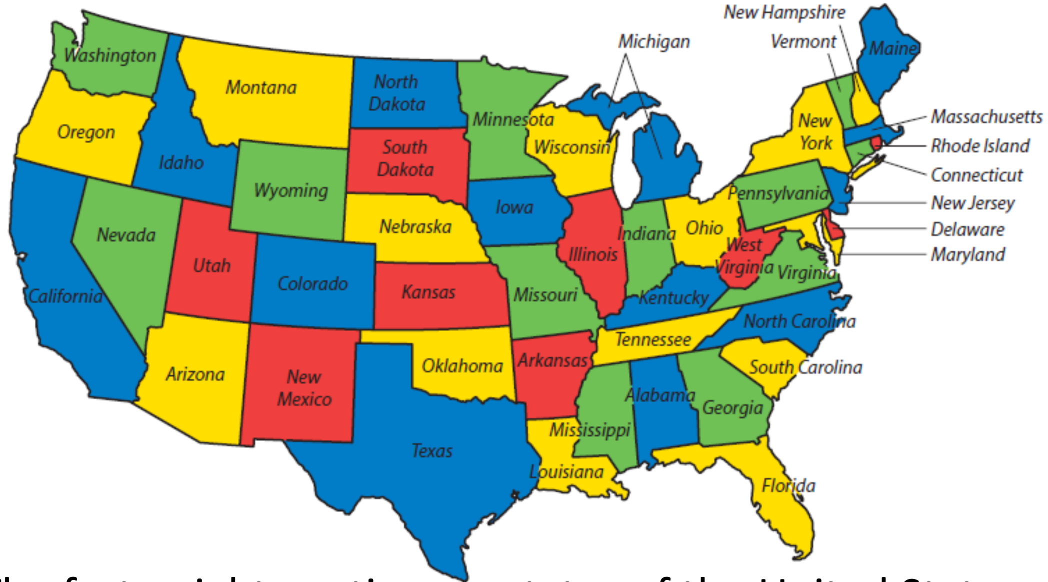
- The background – four color problem
- The Heawood map and the Errera map
- A novel method of rotation
- Color the Heawood map using the rotation method
- Feature of the Errera map – rotational symmetry
- Color the Errera map using the multi-start rotation method
- Conclusions

The background

- The four-color problem was first posed by Francis Guthrie in 1852 [2].
- No more than four colors are required to color the regions of any map so that no two adjacent regions have the same color.



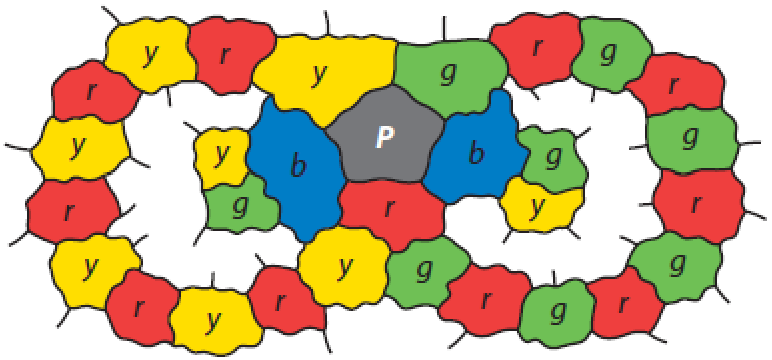
Belgium, France, Germany, and Luxembourg are all neighboring countries, so the map requires four colors [2]



The forty-eight contiguous states of the United States (excluding Alaska and Hawaii) were colored with four colors [2]

The background

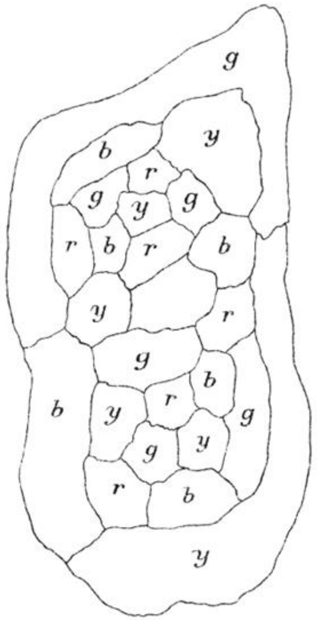
- It can be proved that every map has at least one region with five or fewer neighbors by using Euler's formula [3].
- One proposed proof was given by Alfred Kempe using Kempe Chain in 1879 [2].

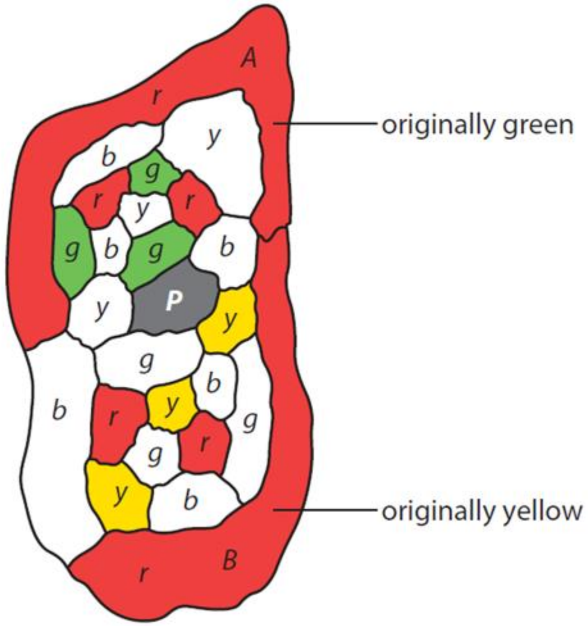


Alfred Bray Kempe (1849–1922)

The background

- Percy Heawood found counterexample of Kempe’s proof in 1890 [2].



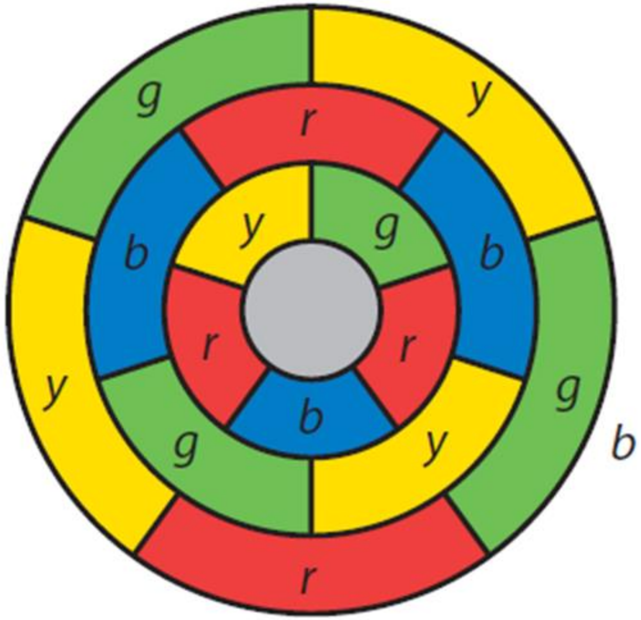


Percy John Heawood (1861–1955)

However, he used the Kempe’s ideas to prove the Five-Color-Theorem.

The background

- Alfred Errera found counterexample of Kempe’s proof in 1921 [1].



A Novel Method of Rotation

- Xie has developed a novel method of rotation inspired by a rotation principle from Zhuan Falun book [4] of Falun Dafa to prove the Four Color Theorem [5].

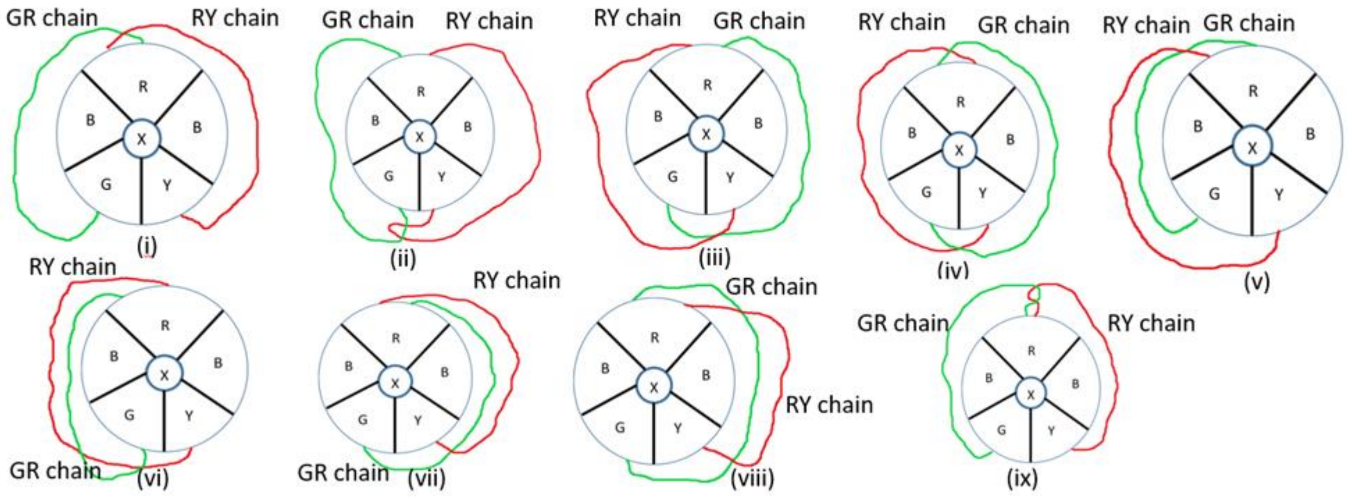


Table 1. The left rotation table with the first 17 rotates [5, 6, 7]. (*Note: after 16 rotates, the relative color and location of the 5-neighbor regions back to the initial one, however, we can just keep increasing the number of the subscript for more rotates.)

Rotates	Chain 1	Chain 2	Share	Doubles	interchange	Form new chains
1	GR	RY	R	B, B	Inside RG, B→Y_1	BG
2	BG	RG	G	Y, Y_1	Inside BG, Y→R_1	Y_1B
3	Y_1B	BG	B	R, R_1	Inside Y_1B, R→G_1	R_1Y_1
4	R_1Y_1	Y_1B	Y_1	G, G_1	Inside R_1Y_1, G→B_1	G_1R_1
5	G_1R_1	R_1Y_1	R_1	B, B_1	Inside G_1R_1, B→Y_2	B_1G_1
6	B_1G_1	G_1R_1	G_1	Y_1, Y_2	Inside B_1G_1, Y_1→R_2	Y_2B_1
7	Y_2B_1	B_1G_1	B_1	R_1, R_2	Inside Y_2B_1, R_1→G_2	R_2Y_2
8	R_2Y_2	Y_2B_1	Y_2	G_1, G_2	Inside R_2Y_2, G_1→B_2	G_2R_2
9	G_2R_2	R_2Y_2	R_2	B_1, B_2	Inside G_2R_2, B_1→Y_3	B_2G_2
10	B_2G_2	G_2R_2	G_2	Y_2, Y_3	Inside B_2G_2, Y_2→R_3	Y_3B_2
11	Y_3B_2	B_2G_2	B_2	R_2, R_3	Inside Y_3B_2, R_2→G_3	R_3Y_3
12	R_3Y_3	Y_3B_2	Y_3	G_2, G_3	Inside R_3Y_3, G_2→B_3	G_3R_3
13	G_3R_3	R_3Y_3	R_3	B_2, B_3	Inside G_3R_3, B_2→Y_4	B_3G_3
14	B_3G_3	G_3R_3	G_3	Y_3, Y_4	Inside B_3G_3, Y_3→R_4	Y_4B_3
15	Y_4B_3	B_3G_3	B_3	R_3, R_4	Inside Y_4B_3, R_3→G_4	R_4Y_4
16	R_4Y_4	Y_4B_3	Y_4	G_3, G_4	Inside R_4Y_4, G_3→B_4	G_4R_4
17	G_4R_4	R_4Y_4	R_4	B_3, B_4	Inside G_4R_4, B_3→Y_5	B_4G_4

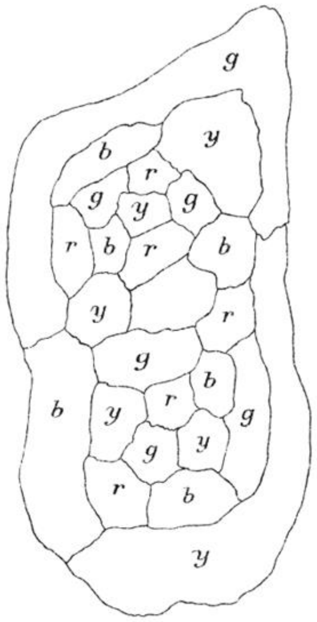
A Novel Method of Rotation

Table 2. The right rotation table with the first 17 rotates (*Note: after 16 rotates, the relative color and location of the 5-neighbor regions back to the initial one, however, we can just keep increasing the number of the subscript for more rotates.)

Rotates	Chain 1	Chain 2	Share	Doubles	interchange	Form new chains
1	GR	RY	R	B, B	Inside RY, $B \rightarrow G_1$	YB
2	RY	YB	Y	G, G_1	Inside YB, $G \rightarrow R_1$	BG_1
3	YB	BG_1	B	R, R_1	Inside BG_1, $R \rightarrow Y_1$	G_1R_1
4	BG_1	G_1R_1	G_1	Y, Y_1	Inside G_1R_1, $Y \rightarrow B_1$	R_1Y_1
5	G_1R_1	R_1Y_1	R_1	B, B_1	Inside R_1Y_1, $B_1 \rightarrow G_2$	Y_1B_1
6	R_1Y_1	Y_1B_1	Y_1	G_1, G_2	Inside Y_1B_1, $G_1 \rightarrow R_2$	B_1G_2
7	Y_1B_1	B_1G_2	B_1	R_1, R_2	Inside B_1G_2, $R_1 \rightarrow Y_2$	G_2R_2
8	B_1G_2	G_2R_2	G_2	Y_1, Y_2	Inside G_2R_2, $Y_1 \rightarrow B_2$	R_2Y_2
9	G_2R_2	R_2Y_2	R_2	B_1, B_2	Inside R_2Y_2, $B_2 \rightarrow G_3$	Y_2B_2
10	R_2Y_2	Y_2B_2	Y_2	G_2, G_3	Inside Y_2B_2, $G_2 \rightarrow R_3$	B_2G_3
11	Y_2B_2	B_2G_3	B_2	R_2, R_3	Inside B_2G_3, $R_2 \rightarrow Y_3$	G_3R_3
12	B_2G_3	G_3R_3	G_3	Y_2, Y_3	Inside G_3R_3, $Y_2 \rightarrow B_3$	R_3Y_3
13	G_3R_3	R_3Y_3	R_3	B_2, B_3	Inside R_3Y_3, $B_3 \rightarrow G_4$	Y_3B_3
14	R_3Y_3	Y_3B_3	Y_3	G_3, G_4	Inside Y_3B_3, $G_3 \rightarrow R_4$	B_3G_4
15	Y_3B_3	B_3G_4	B_3	R_3, R_4	Inside B_3G_4, $R_3 \rightarrow Y_4$	G_4R_4
16	B_3G_4	G_4R_4	G_4	Y_3, Y_4	Inside G_4R_4, $Y_3 \rightarrow B_4$	R_4Y_4
17	G_4R_4	R_4Y_4	R_4	B_3, B_4	Inside R_4Y_4, $B_4 \rightarrow G_5$	Y_4B_4

Heawood’s map

- This historic Heawood map has 25 regions, the full combinations for every region possible to have one of four colors will be a very large number as 4^{25}.
- It can be very challenging to just use trial and error method to make it four-colored, even people with some expertise of graph coloring may still need considerable time to do it.
- Therefore, it is necessary to develop a systematic way to achieve it, and the systematic way will have potential wider applications in coloring graphs, especially with a much larger number of regions in a map.



Color the Heawood map using the rotation method

- Preparation

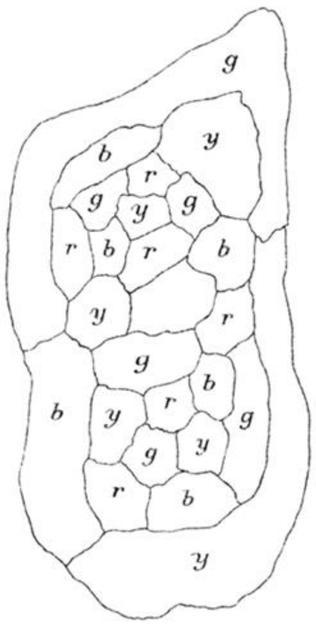


Figure 1

All color r-b interchange

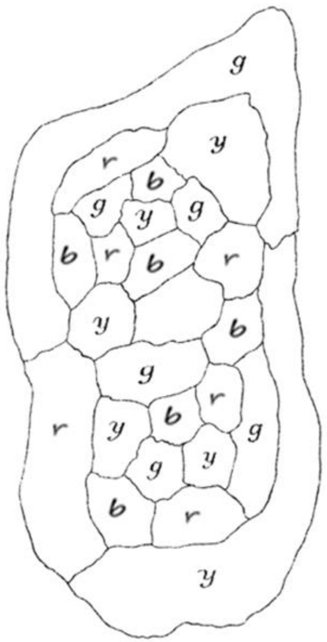


Figure 2

All color y-g interchange

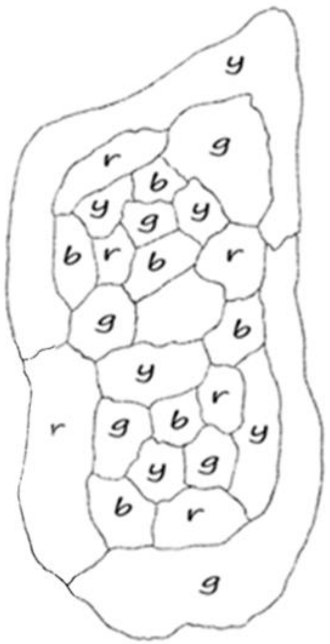


Figure 3

To open the left-side boundary of r (red) region to make the region fully cover a sphere surface

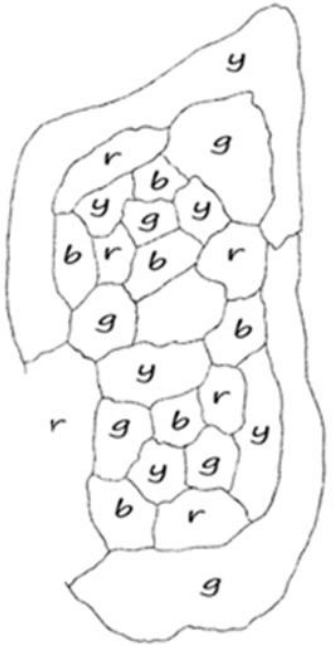


Figure 4

Color the Heawood map using the rotation method

- To mark the RY chain and GR chain

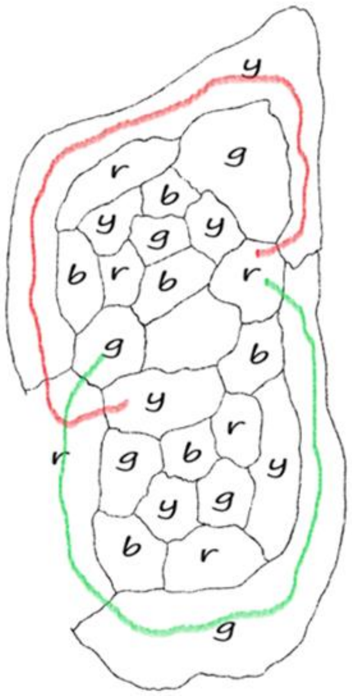


Figure 5

Color the Heawood map using the rotation method

- To follow the Step 1 rotation in Table 1 to perform B→Y_1 chain color interchange inside GR chain in Figure 5 (*note: the meaning inside GR chain is the side of g(green)-b(blue)-r(red) of the five neighbor regions, rather than the side of g(green)-y(yellow)-b(blue) –r(red) of the five neighbor regions).

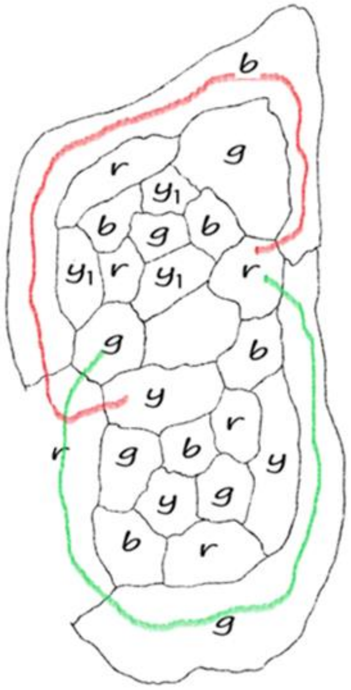


Figure 6

Color the Heawood map using the rotation method

- After the B→Y_1 chain color interchange, to check whether the new chain BG (in the last column of the row 1 in the Table 1, which can be just a single region for certain special cases) can be formed. The answer is "Yes" for BG chain being formed for the 5-neighbor regions of the middle uncolored region. The BG chain has been marked and shown in Figure 7.

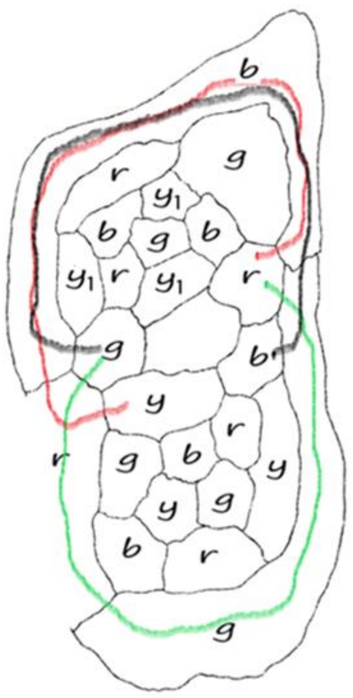


Figure 7

Color the Heawood map using the rotation method

- To follow the Step 2 rotation in Table 1 to perform Y→R_1 chain color interchange inside BG chain in Figure 7 (*note: the meaning inside BG chain is the side of b(blue) - y(yellow) - g(green) of the five neighbor regions, rather than the side of b(blue) – r(red) – y_1(yellow) -g(green) of the five neighbor regions). It becomes Figure 8.

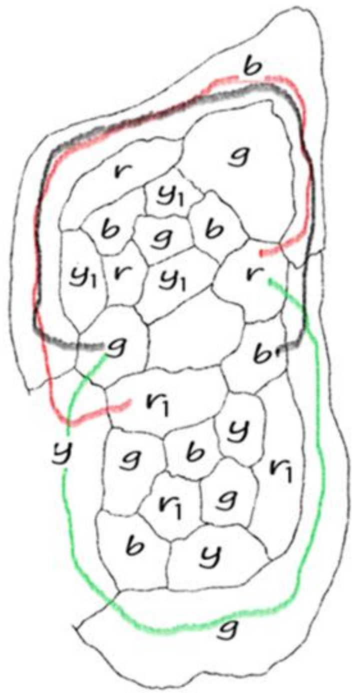


Figure 8

Color the Heawood map using the rotation method

- After the $Y \rightarrow R_1$ chain color interchange, to check whether the new chain Y_1B (in the last column of the row 2 in the Table 1, which can be just a single region for certain special cases) can be formed. The answer is “No” for Y_1B chain being formed for the 5-neighbor regions of the middle uncolored region. The Y_1B chain (and YB chain because Y_1 and Y are just the same yellow color) starting from Y_1 of the 5-neighbor regions of the middle uncolored region has been marked and shown in Figure 9.

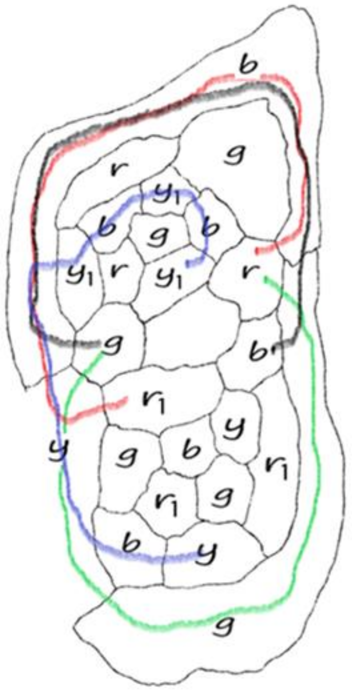


Figure 9

Color the Heawood map using the rotation method

- To perform B-Y_1 chain color interchange, the left-side edge can be closed and back to the original edge of the planar map. Now the uncolored pentagon region can be colored as B (blue). The final result of the historic Heawood map that can be successfully four-colored by this novel rotation method is shown in Figure 10.

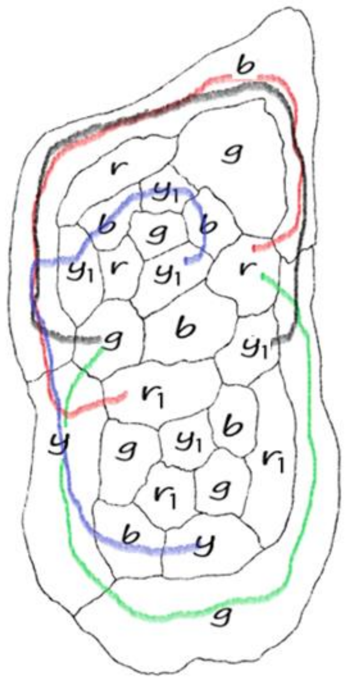
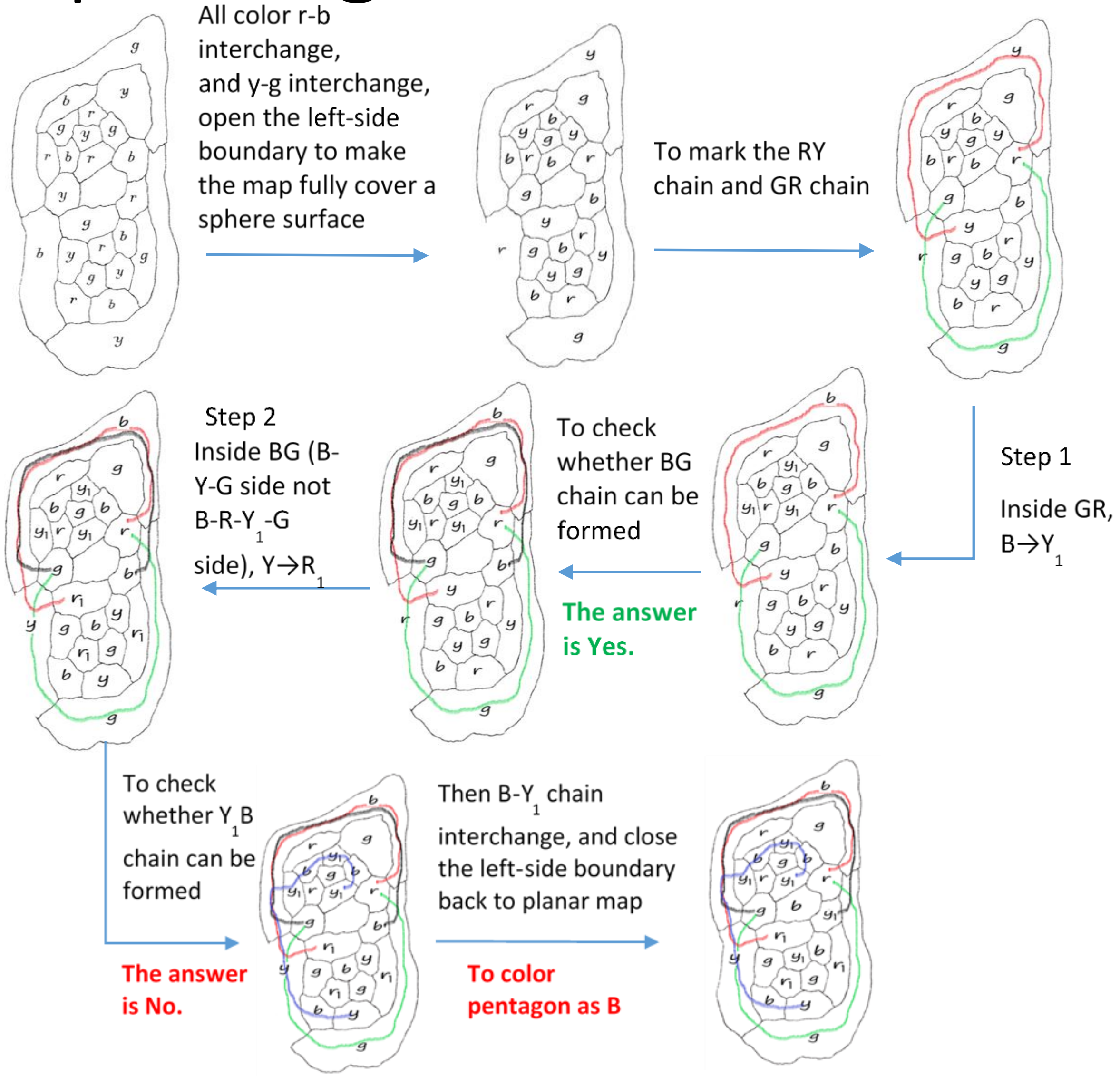


Figure 10

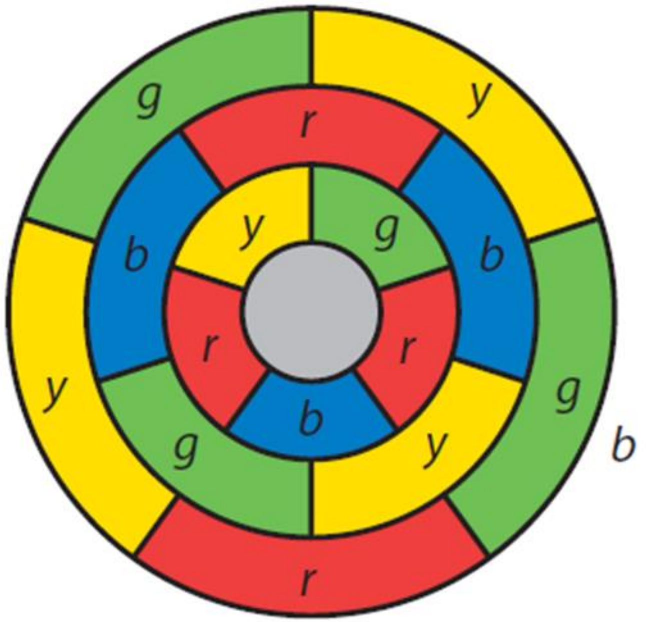
Color the Heawood map using the rotation method

- To put all steps in one slide

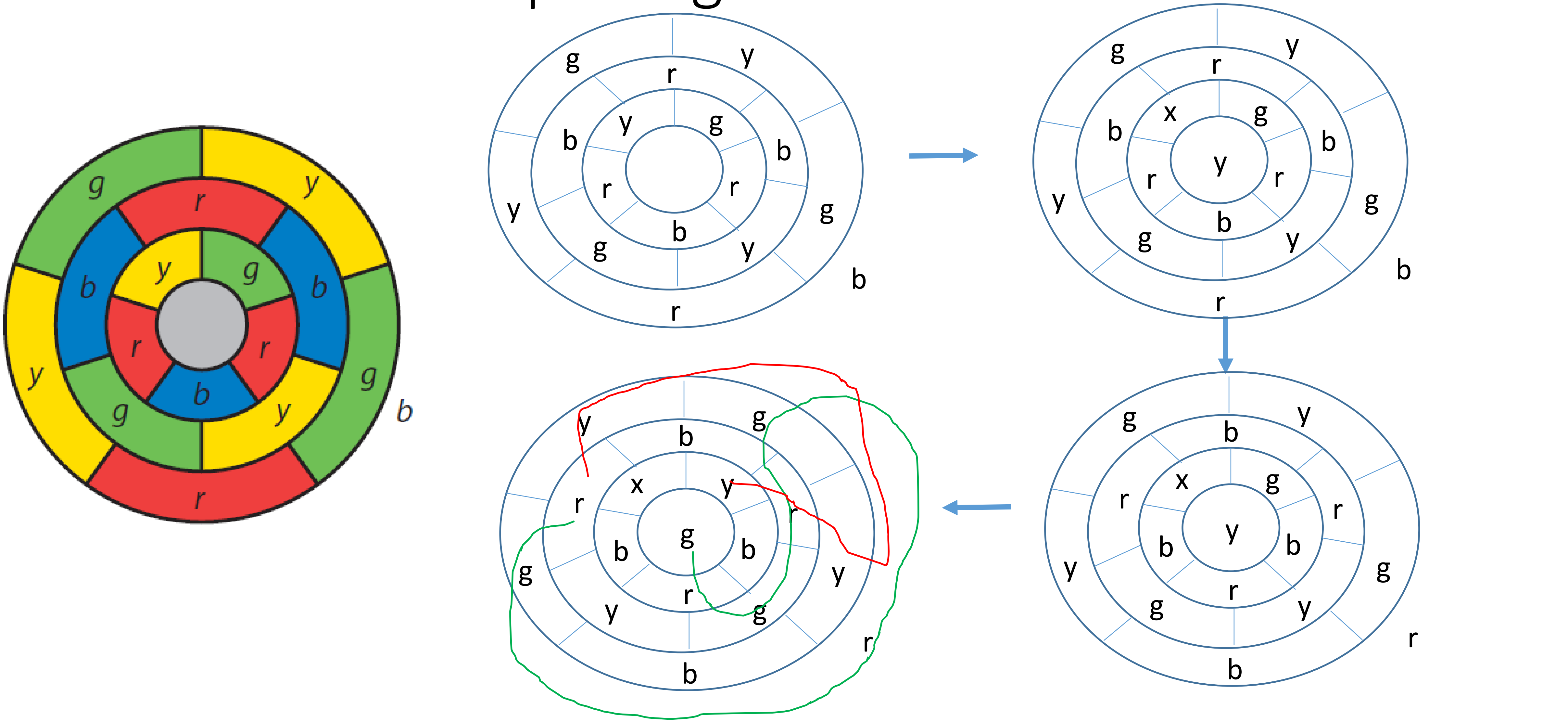


Feature of the Errera map – rotational symmetry

- This Errera map has 17 regions, the full combinations for every region possible to have one of four colors will be a very large number as 4^{17}.
- After 20 rotations, we return to the original coloring.
- Therefore, it is necessary to break the rotational symmetry and then the rotational method can be applied in coloring this kind of graphs. ----Idea of multi-start.



Color the Errera map using the multi-start rotation method



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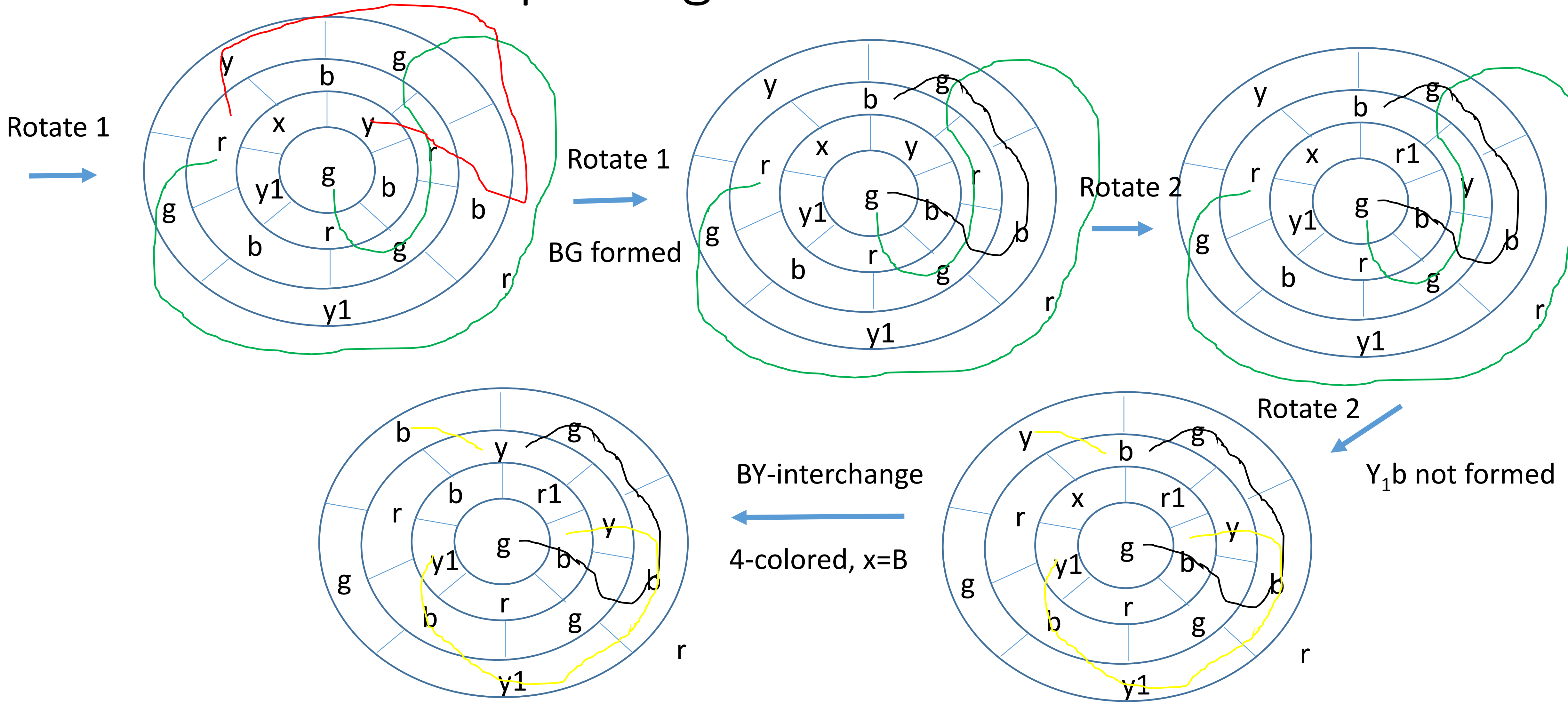
Color the Errera map using the multi-start rotation method



Results

Theorem (Xie, 2022)

The Heawood map can be 4-colored by the rotation method.

Theorem

The Errera map can be 4-colored by the multi-start rotation method.

Conclusions

- A systematic way to color the historic Heawood map with four colors by the novel method of rotation [5] inspired by a rotation principle from Zhuan Falun book [4] of Falun Dafa is shown in this article.
- The rotation method has been applied to color Heawood map and Errera map. Errera map needs to use multi-start to first break the rotational symmetry.
- It shows that the novel method of rotation is very powerful and can provide a systematic approach to make maps four-colored. Running over 30 millions of simulations, this has effectively colored graphs ranging a large variety of number of regions.
- It can have many potential applications in many areas.

References

1. Errera, A. , “Du coloriage des cartes et de quelques questions d’´analysis situs. ” PhD thesis, 1921.
2. Wilson, R., “Four Colors Suffice, Princeton Science Library, Princeton, NJ: Princeton University Press, ISBN 978-0-691-15822-8. (2014).
3. Euler, L., “Elementa doctrinae solidorum”. Novi Commentarii Academiae Scientiarum Petropolitanae: 109–140 (1758).
4. Li, H., “Zhuan Falun” (Revolving The Law Wheel). Publisher: Universe Pub Ny Corp; 2nd edition, ISBN 978-9628143047. (Online version can be downloaded from https://en.falundafa.org/falun-dafa-books.html) (1997).
5. Xie, W., “A Novel Method to Prove the Four Color Theorem,” Authorea. DOI: 10.22541/au.166490983.39190127/v1. (2022).
6. Xie, W., “A Systematic Rotation Method to Color the Historic Heawood Map by Four Colors.” Authorea. DOI: 10.22541/au.166931559.90016969/v1. (2022).
7. Xie, W., “To Color the Historic Heawood Map with Four Colors Using a Systematic Rotation Method,” In 54th Southeastern International Conference on Combinatorics, Graph Theory & Computing (2023).
8. Xie, W., Bowling, A., “To Color the Errera Map and its Variations Using Four Colors. Authorea, DOI: 10.22541/au.168787001.11288089/v1. (2023)

Thank you!