EMI (Exchange of Mathematical Ideas) 2023 Conference, the University of Northern Iowa, August 11-13, 2023

Rotation Operations on the Errera Map and its Variations – Idea II

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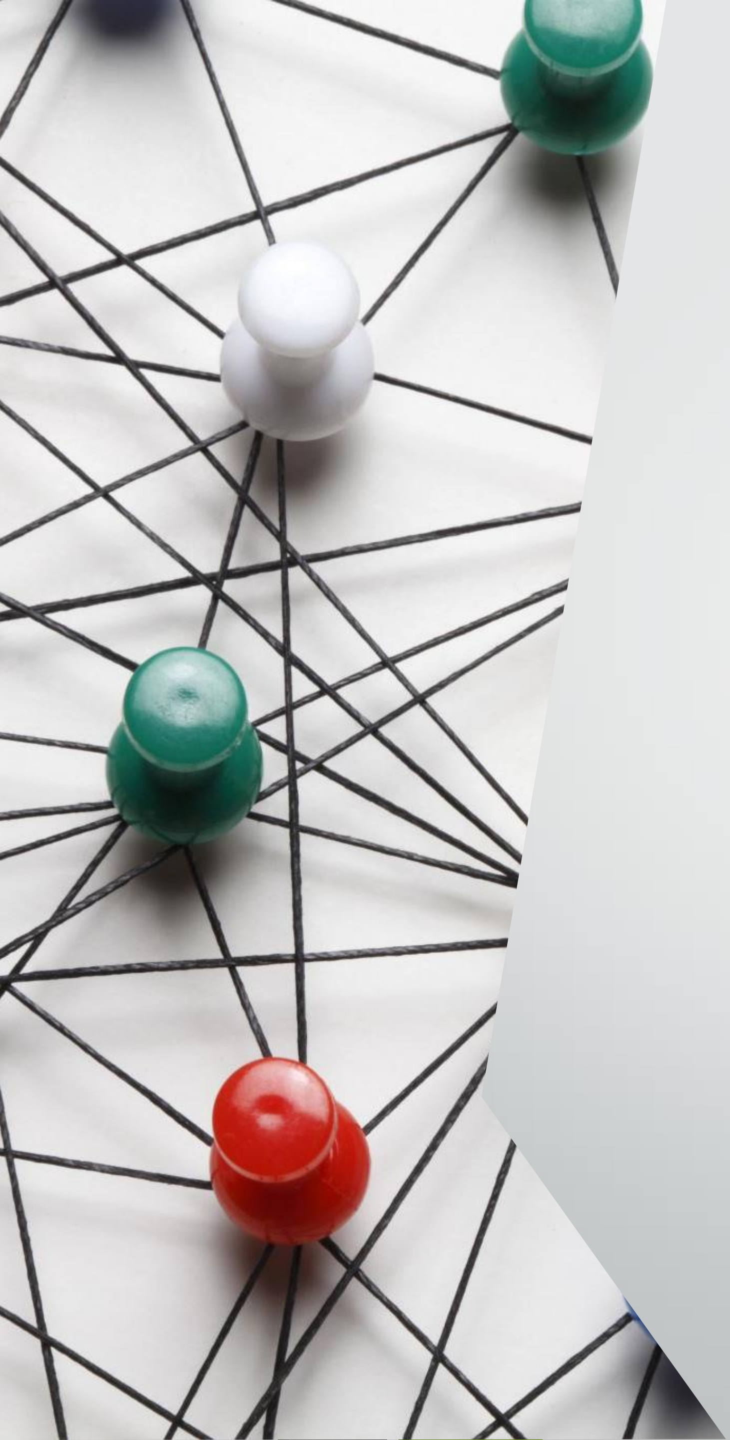

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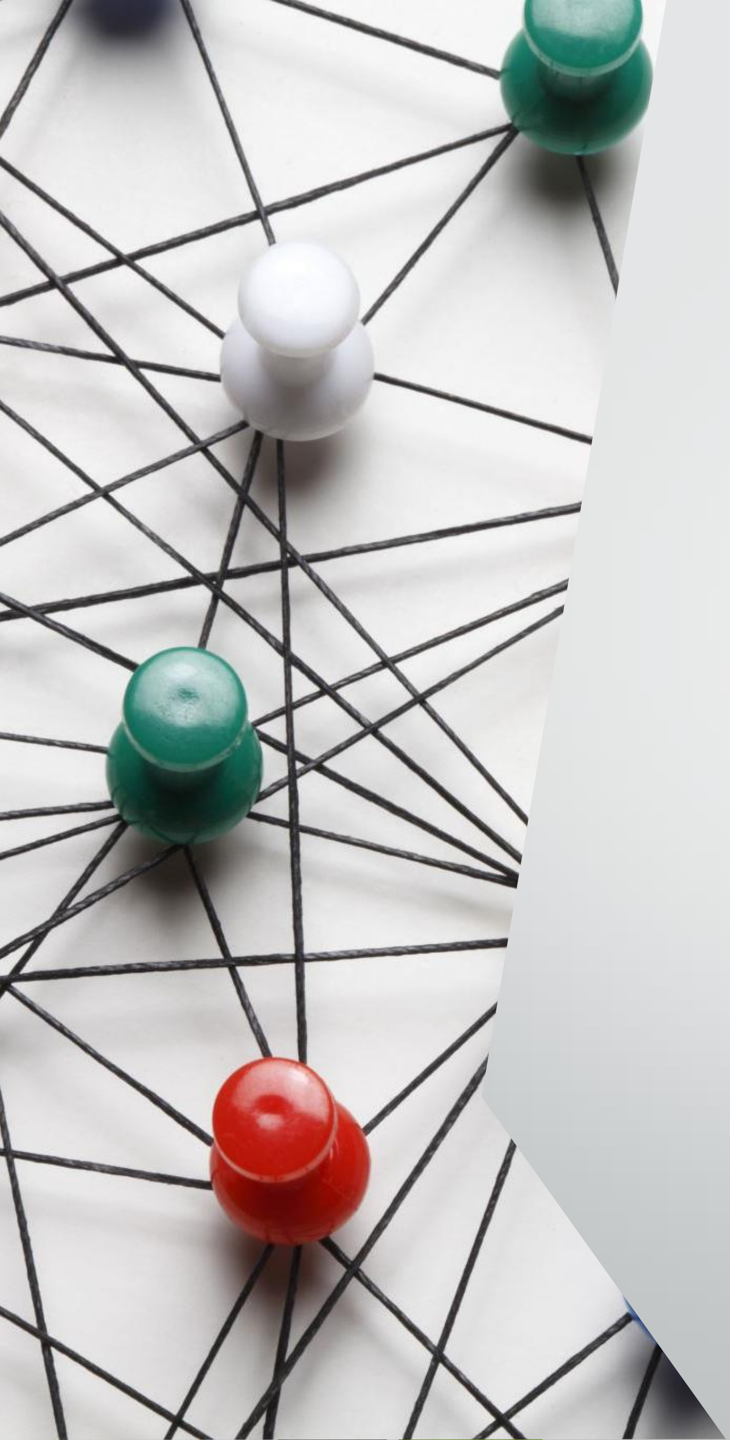

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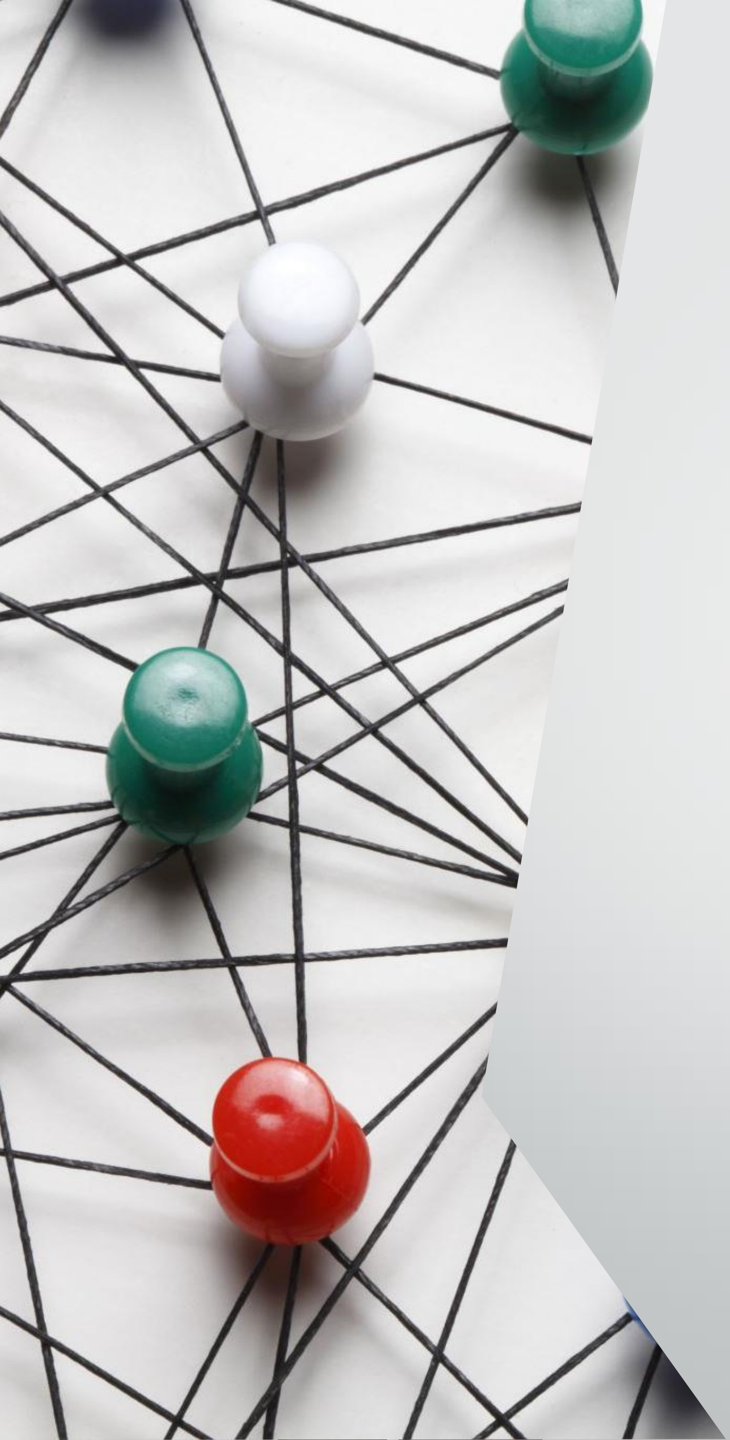

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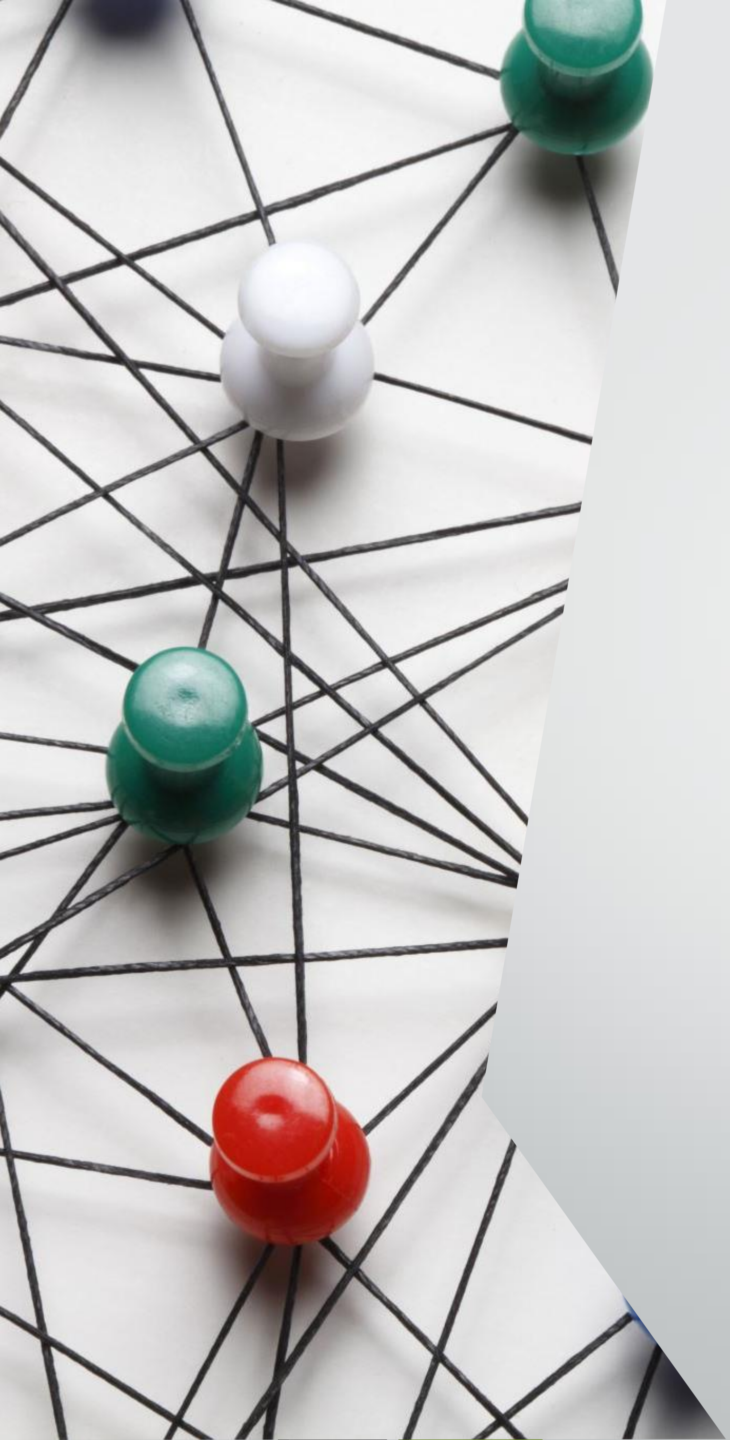

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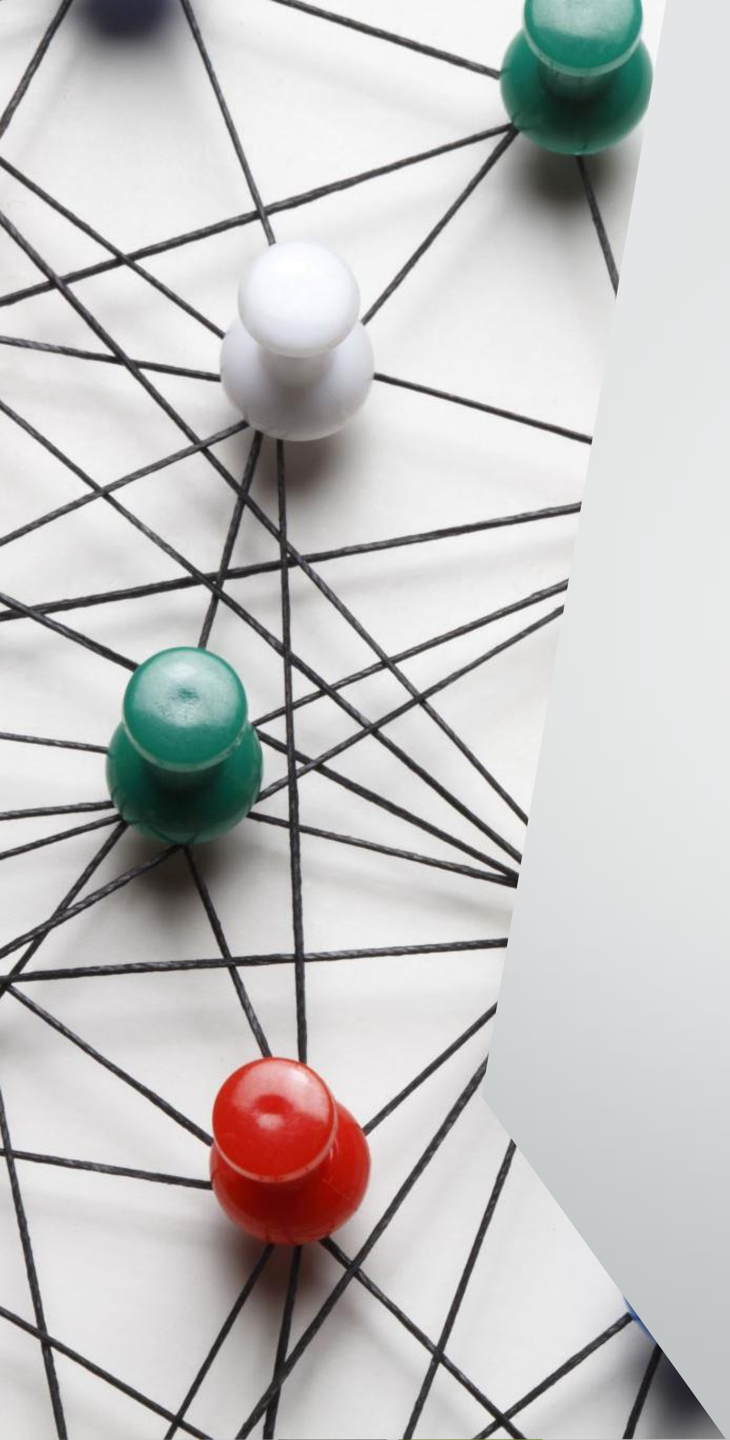

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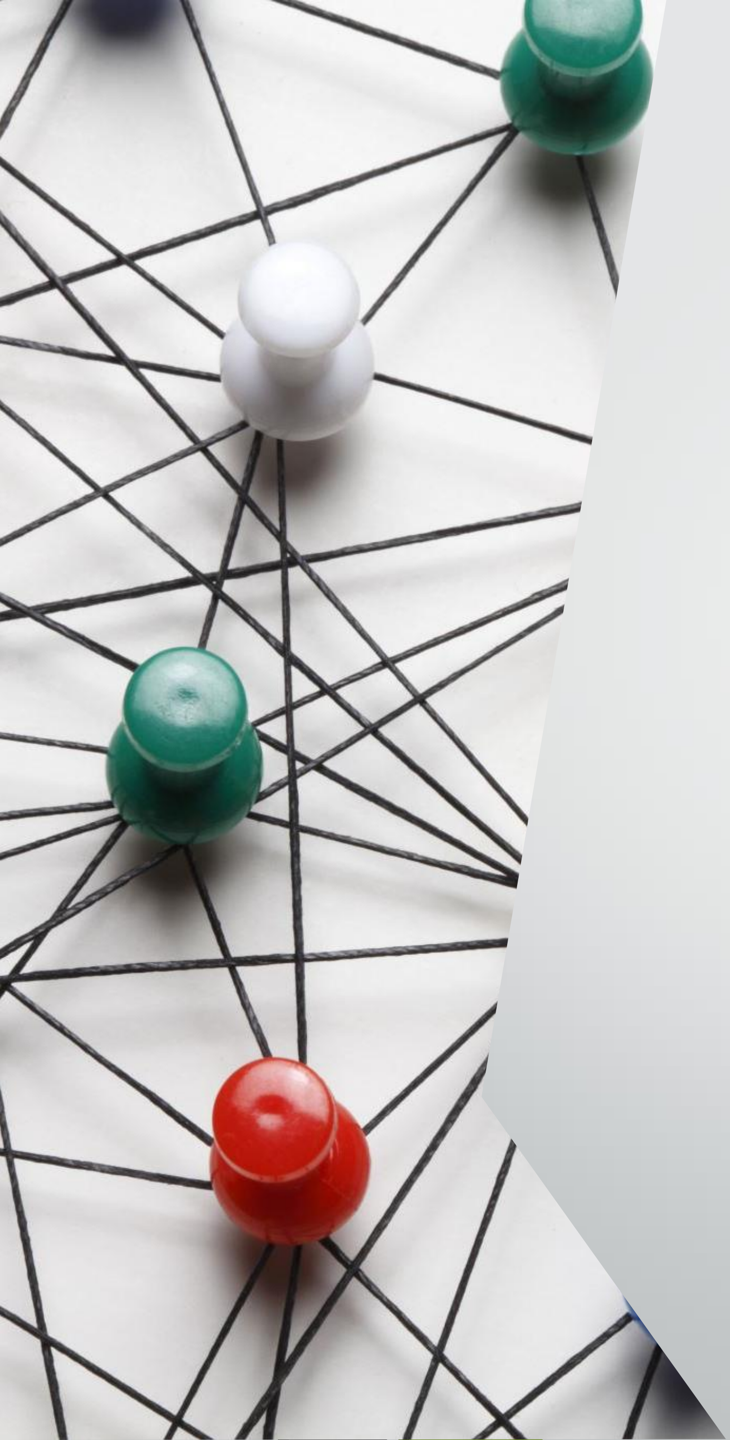

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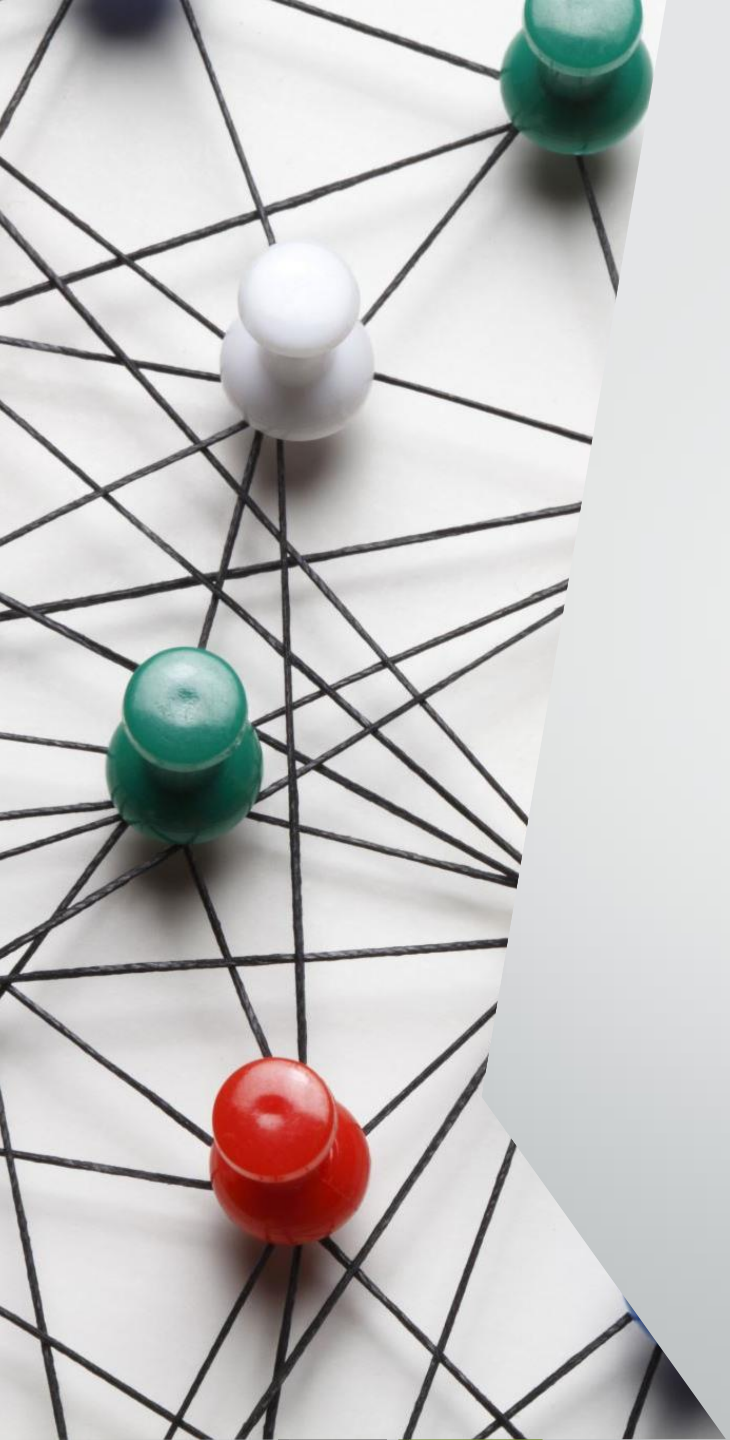

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- Explore the efficacy of a coloring algorithm based on the Rotation Method

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  - This is called a *proper 4 region coloring*, or simply a *4 coloring.*
- **The Four Color Conjecture:** Every plane graph has a proper 4 region coloring.

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- Unfortunately, this proof is heavily based on computers, and even the parts that could theoretically be checked by hand have not been verified by human readers.
- This proof has been simplified by Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas, but the proof still requires the use of computers for verification ([7]).

Question: Can the Four Color Theorem be proven without using computers?

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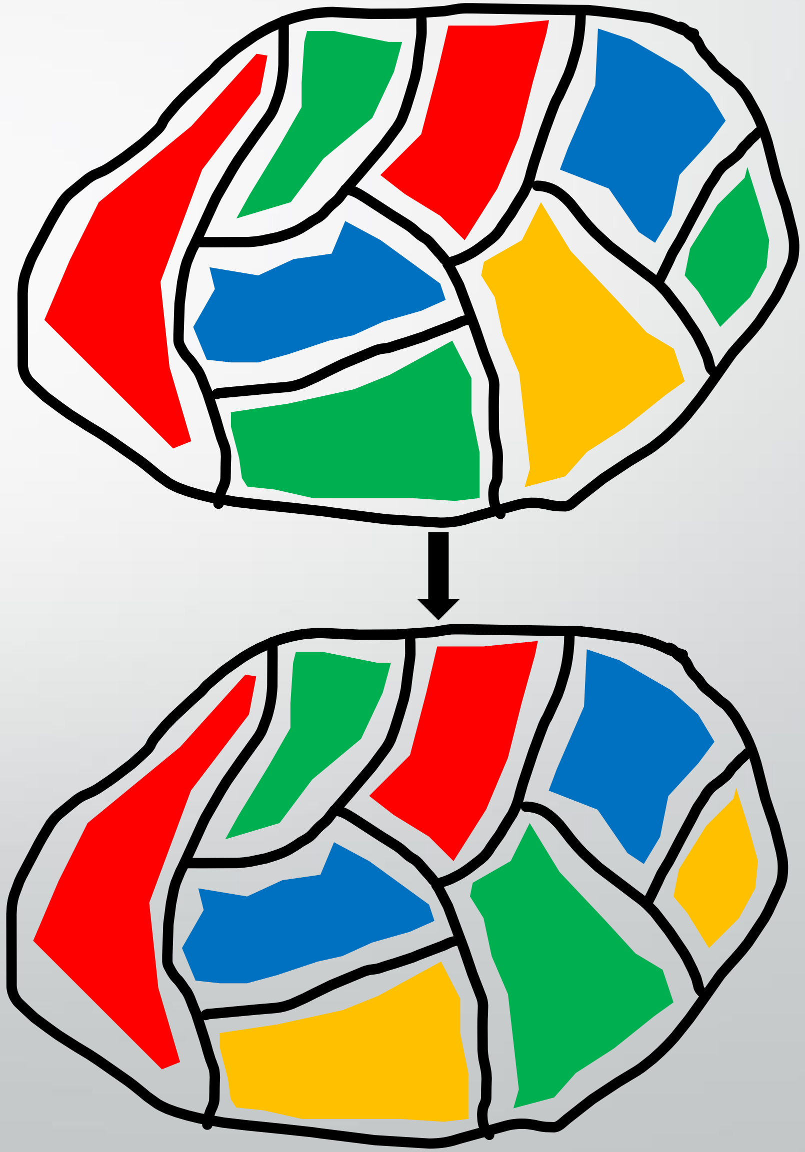

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  - Color all but one region, with the uncolored region having at most 5 neighbors
  - Use various methods (mostly based on Kempe chains) to obtain a coloring of the final region.

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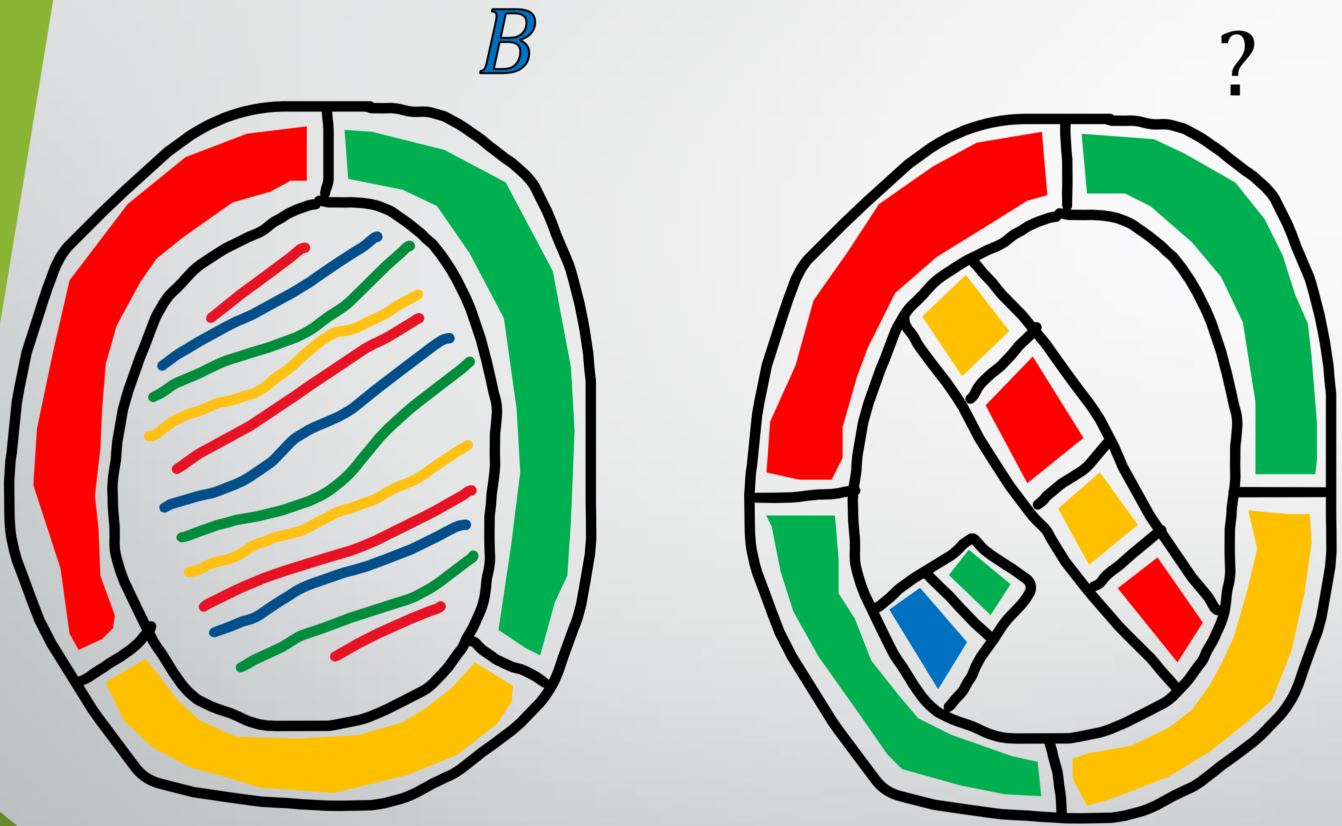

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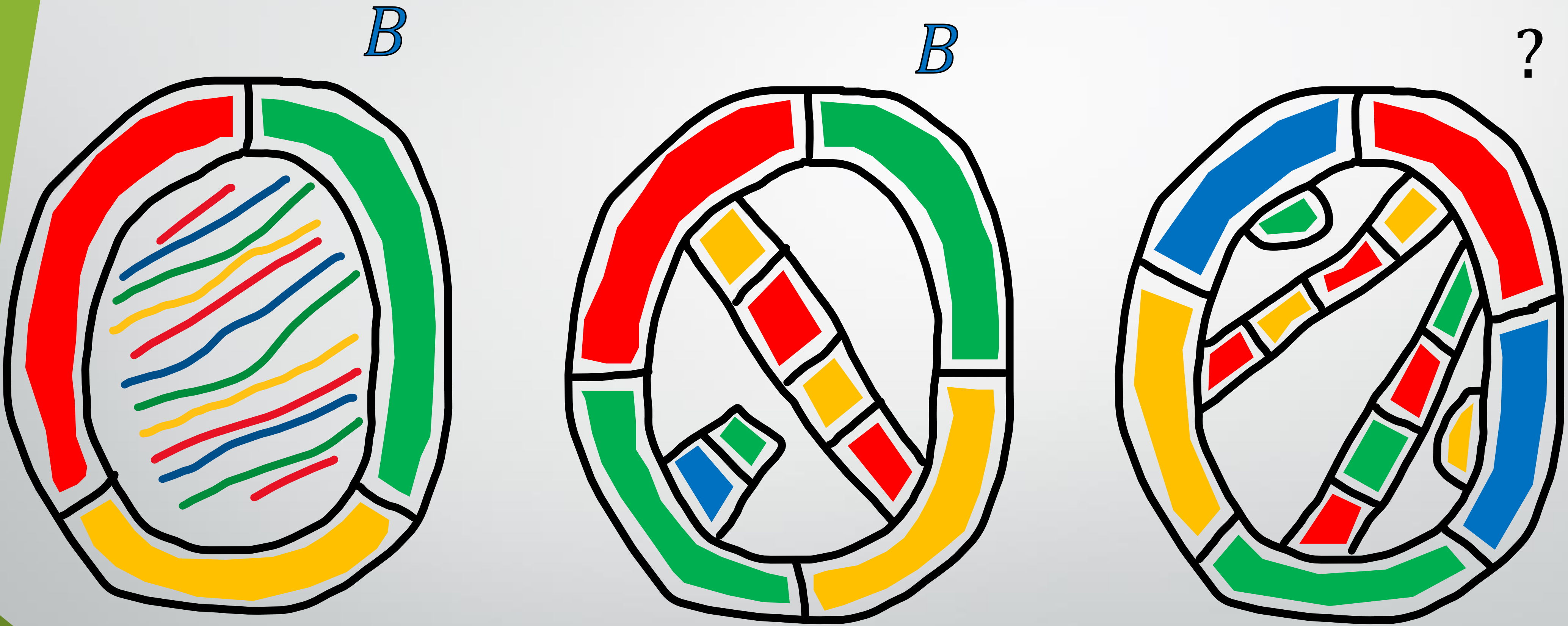

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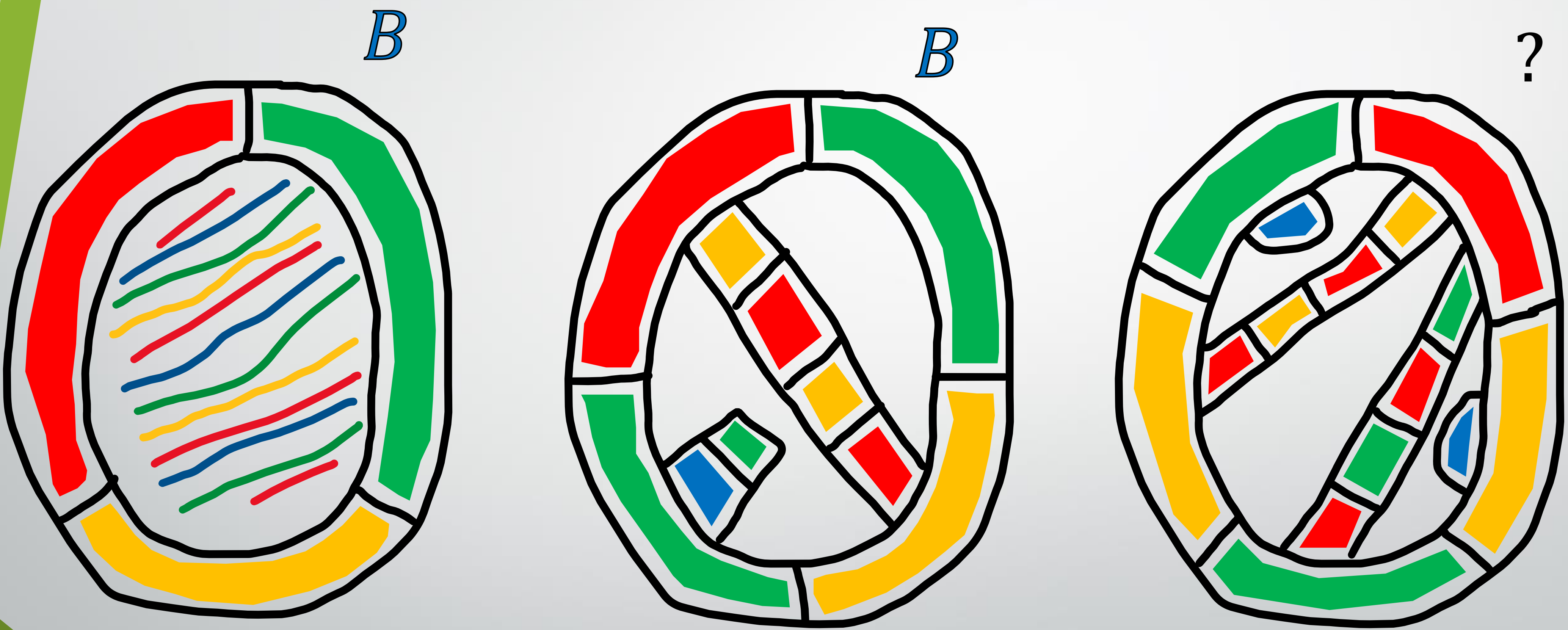

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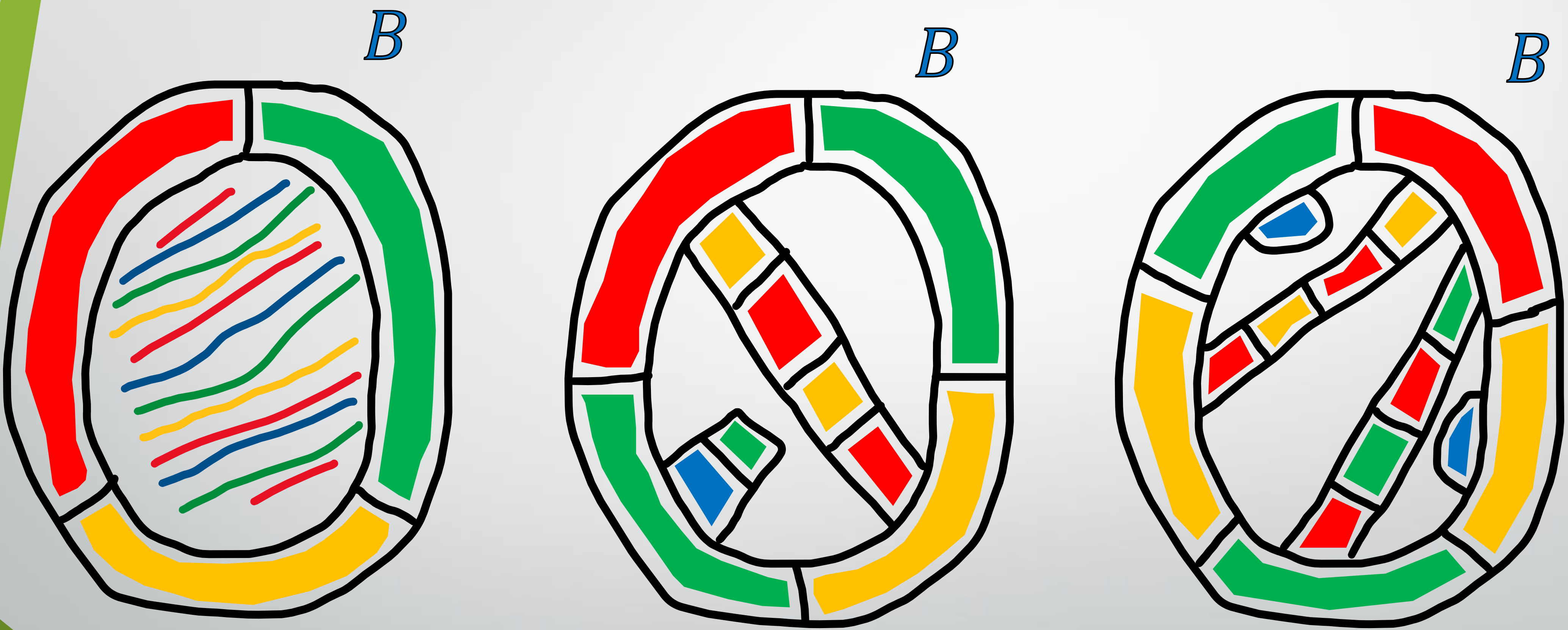
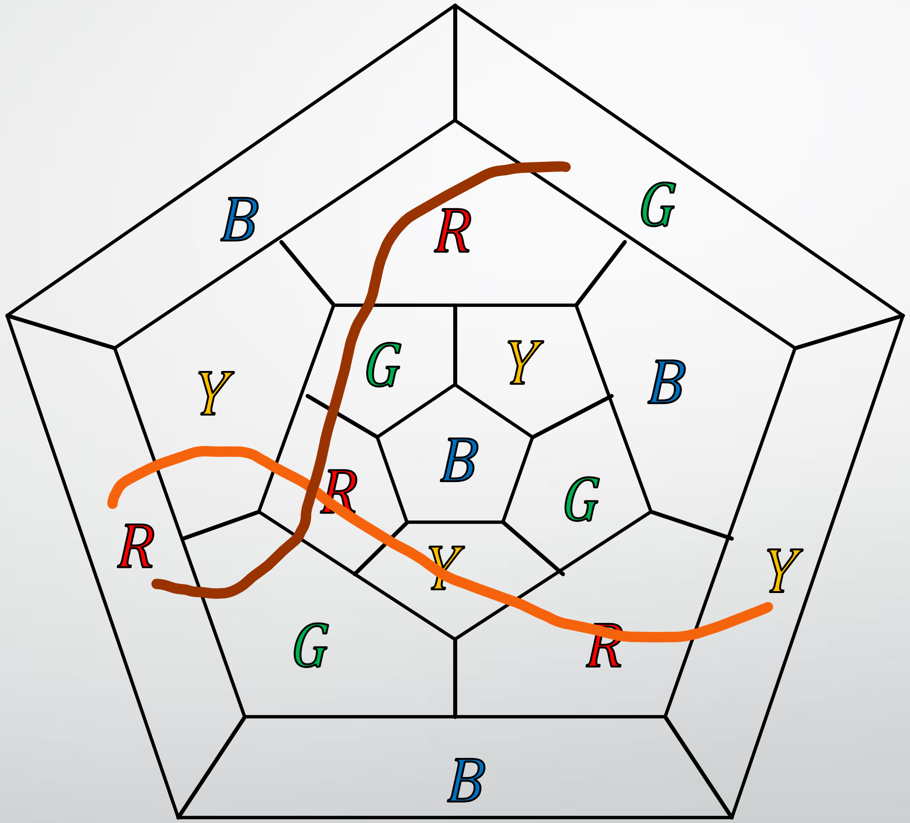


Counterexample to Kempe's Proof



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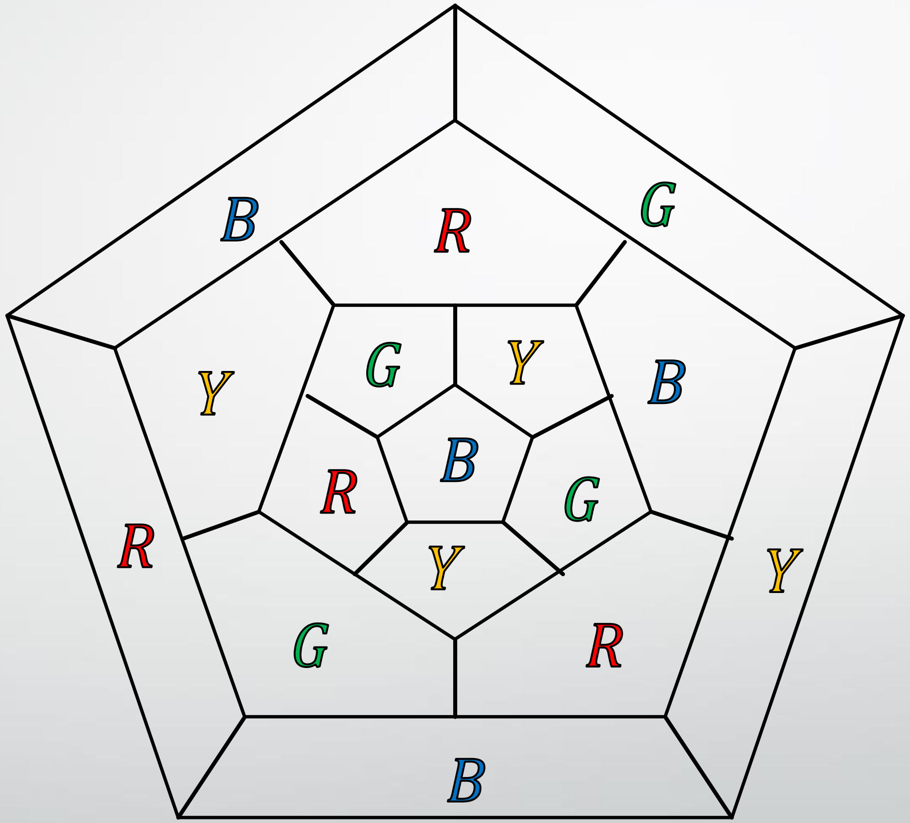

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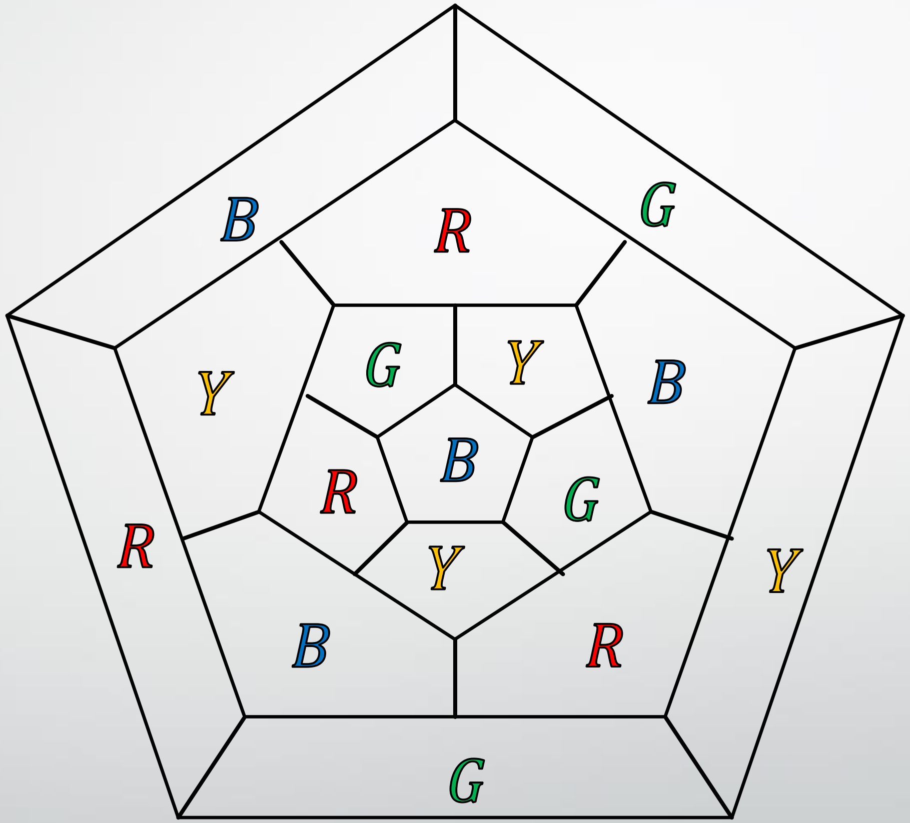

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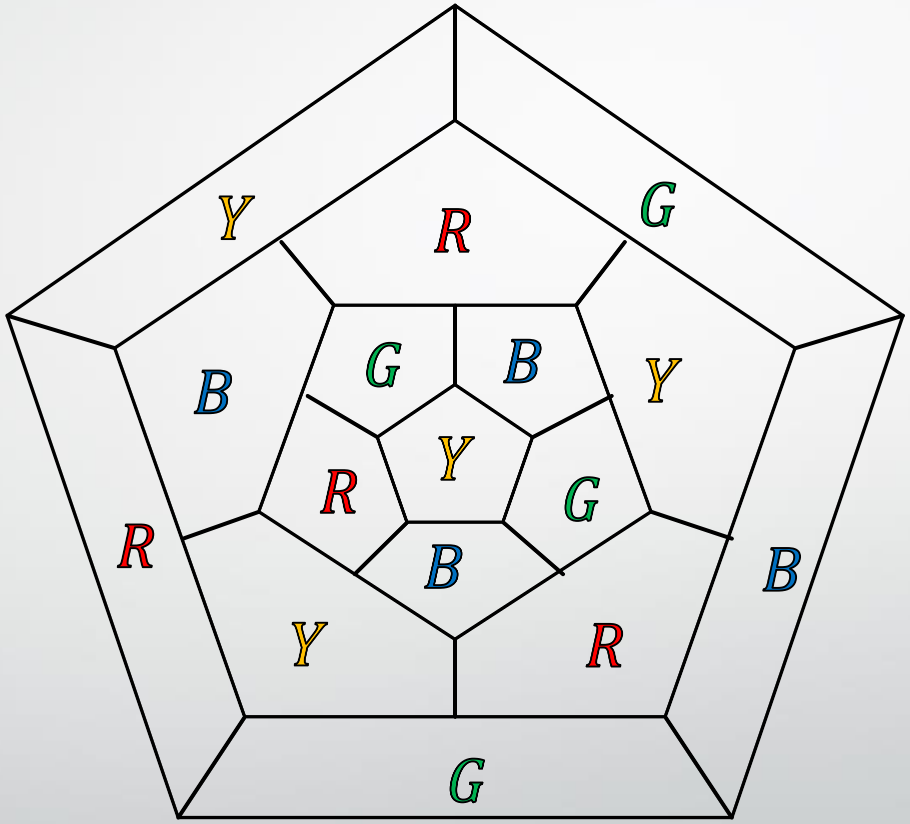

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- The example on the previous slide is due to Alfred Errera in 1921 ([2]). This map has additional special properties, and will be the focus of our study here.

Kittell’s Operations

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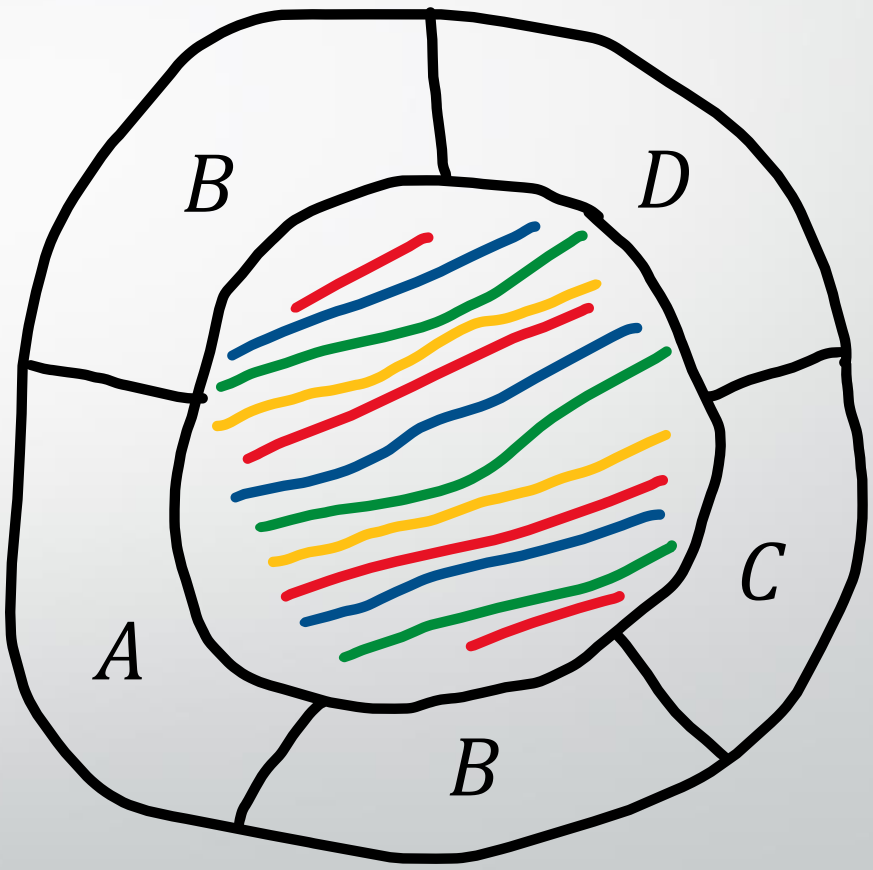
- In his 1935 article “A Group of Operations on a Partially Colored Map,” Irving Kittell studied a variety of different possible Kempe chain color exchanges ([5]).

Definitions



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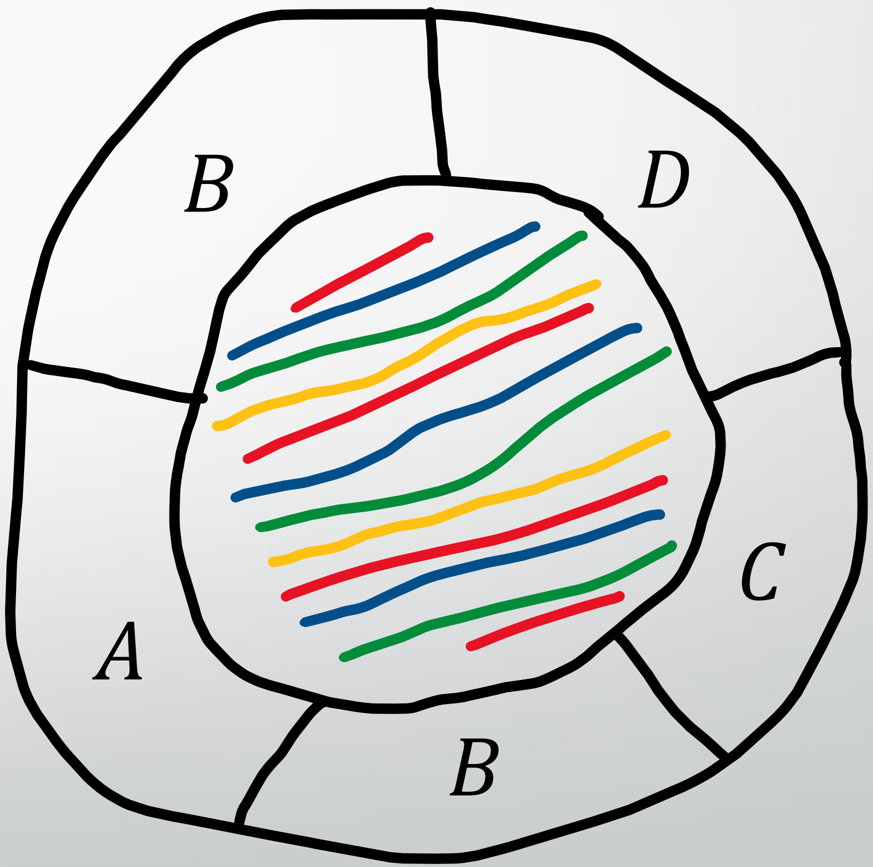

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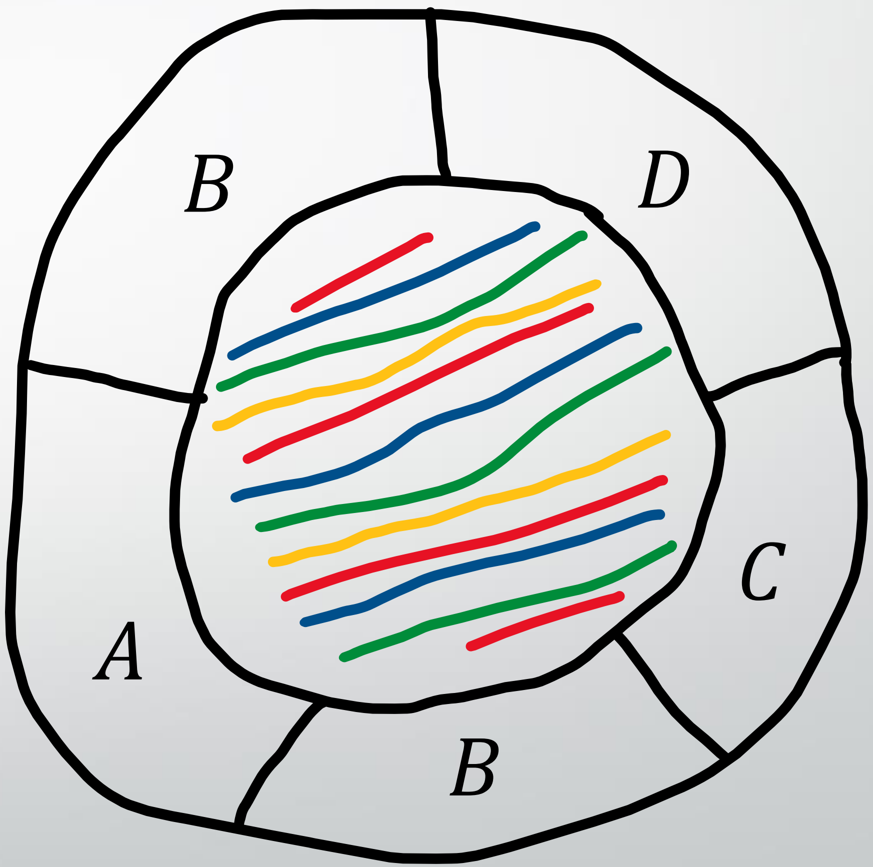

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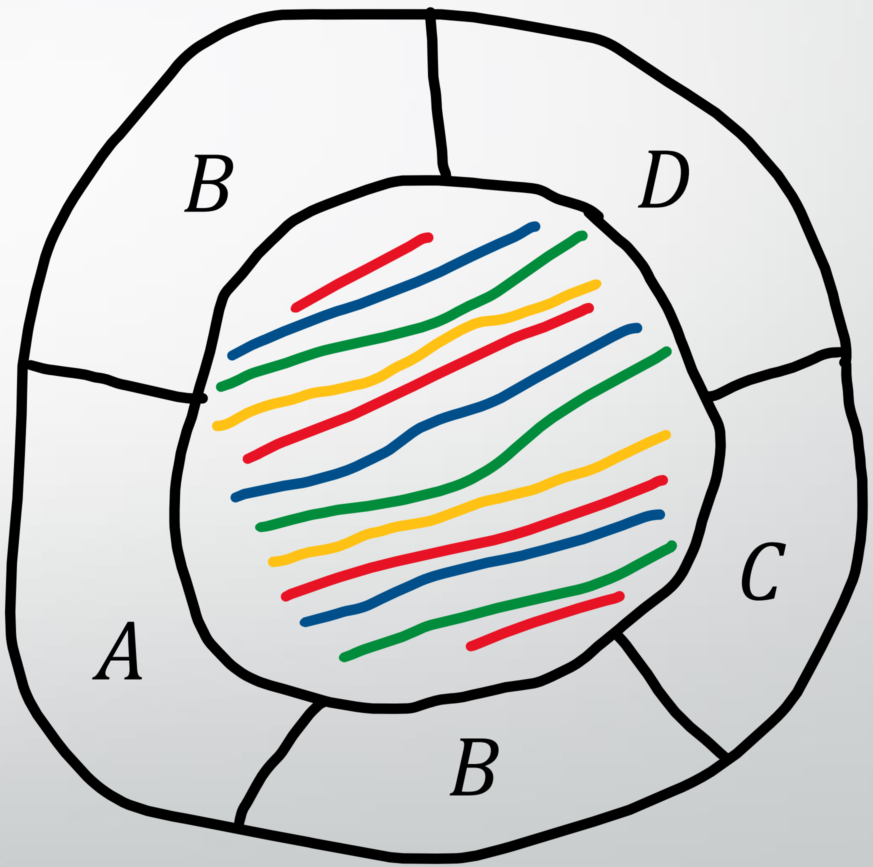

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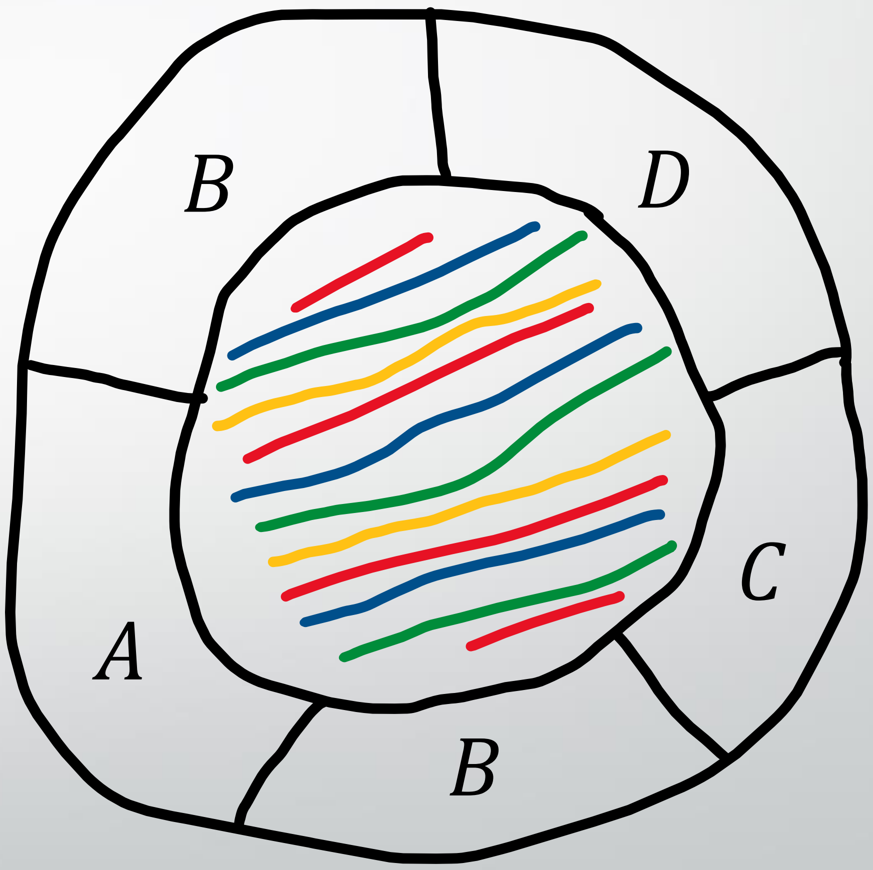

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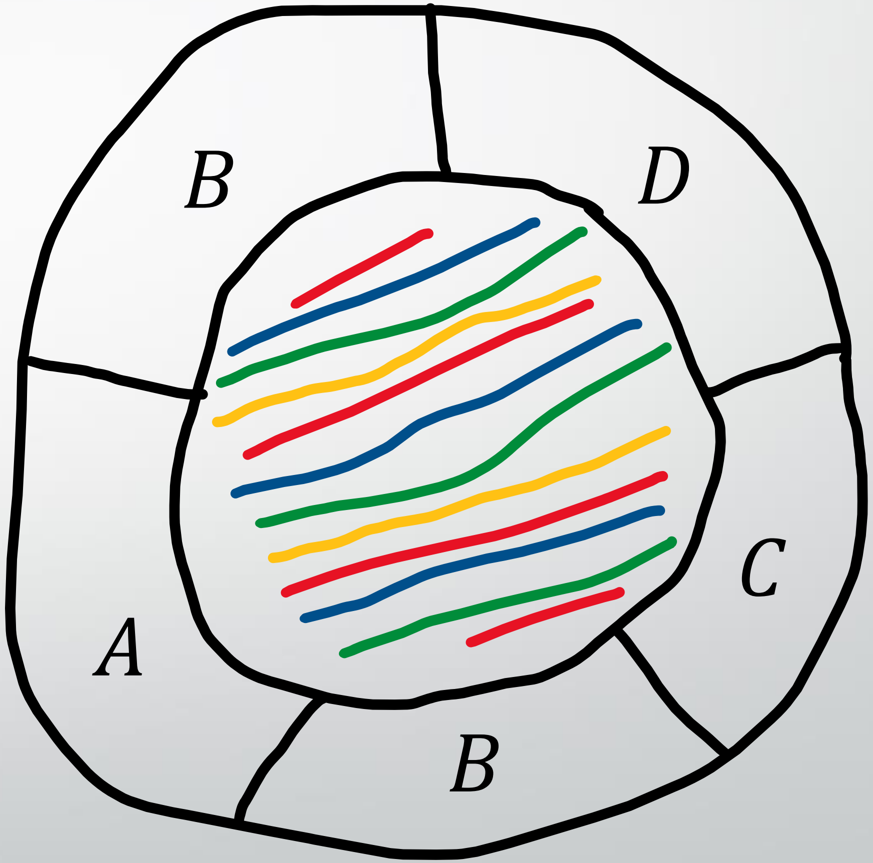

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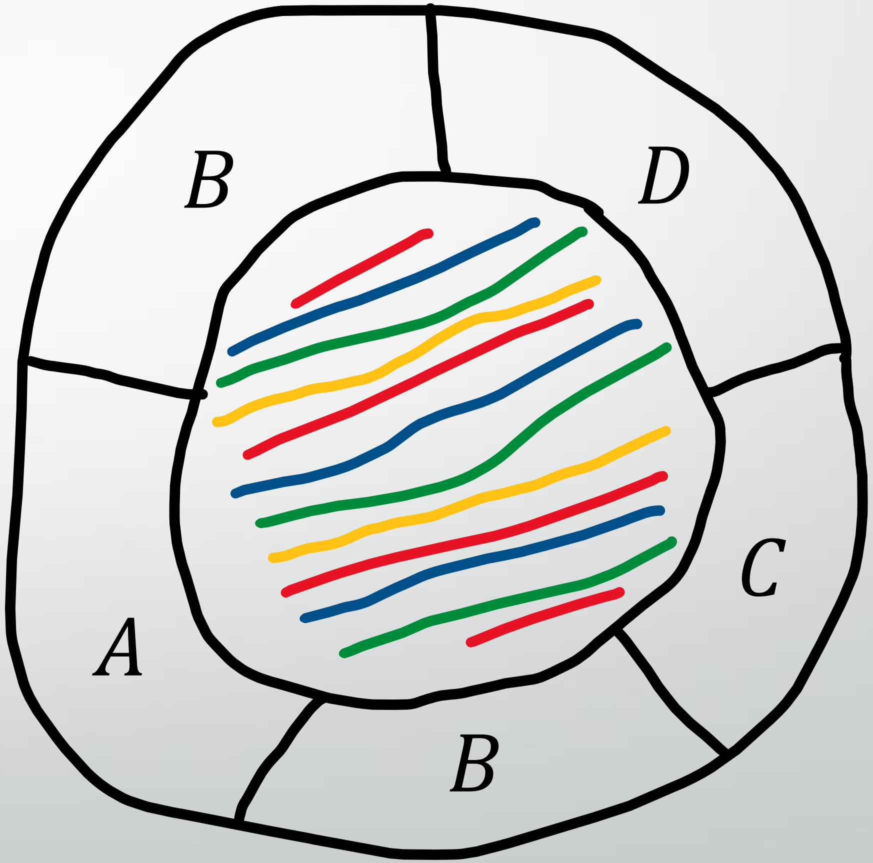
- The CD-Kempe chain is called the *end tangent chain.*



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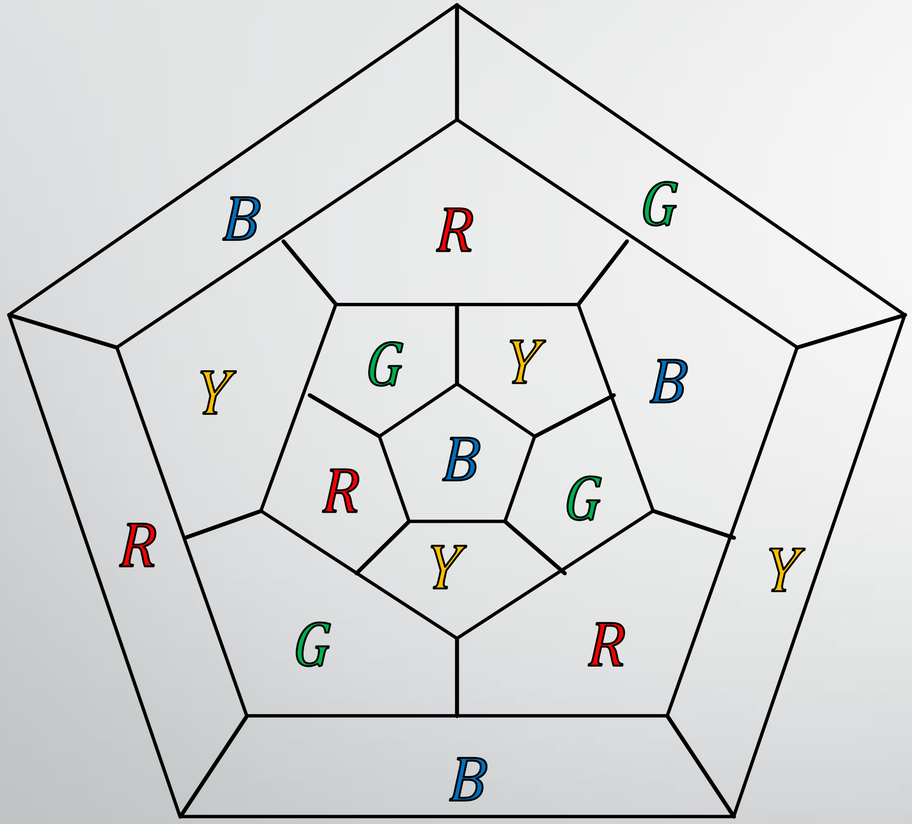

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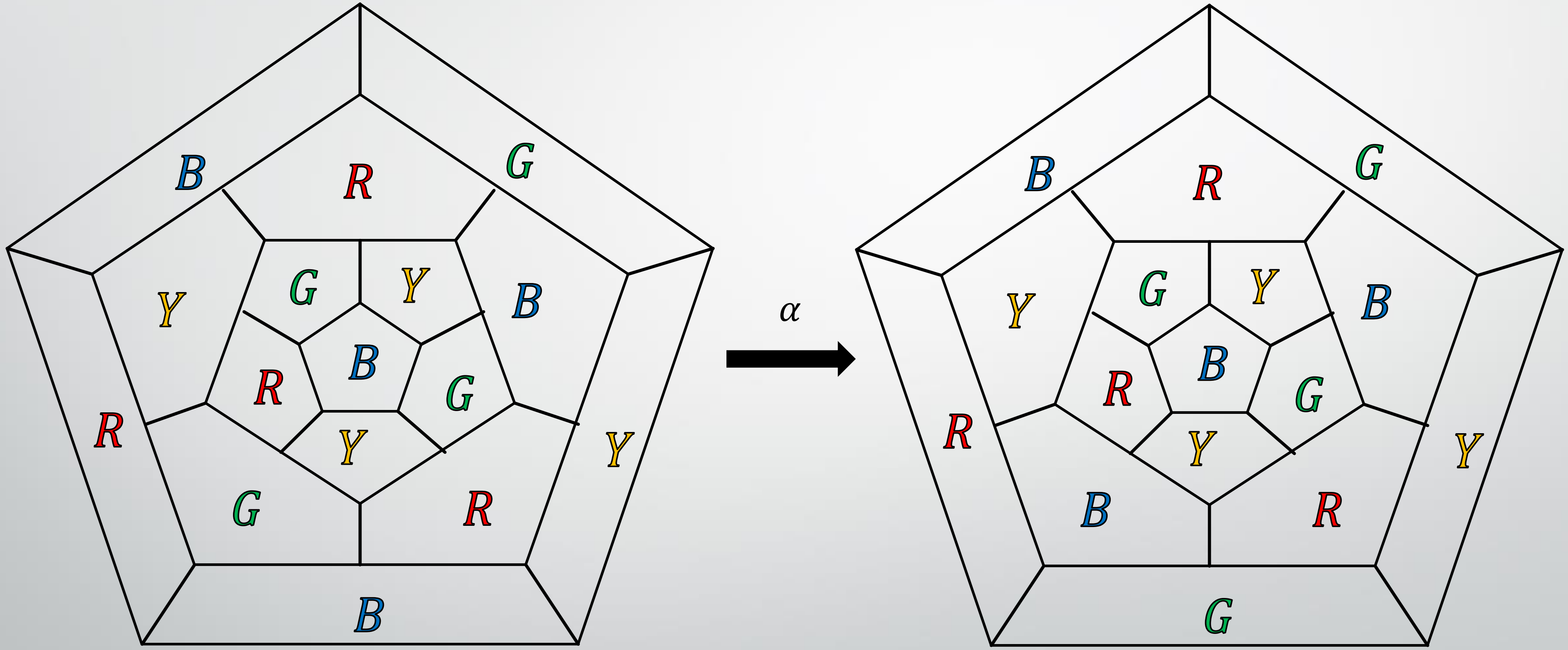

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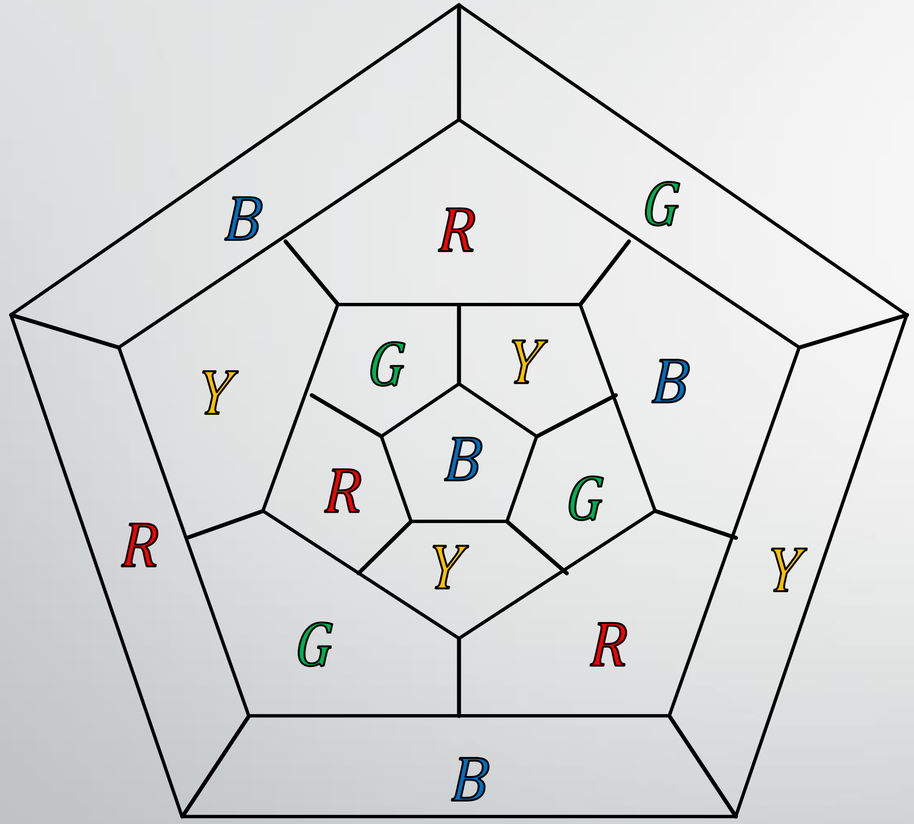

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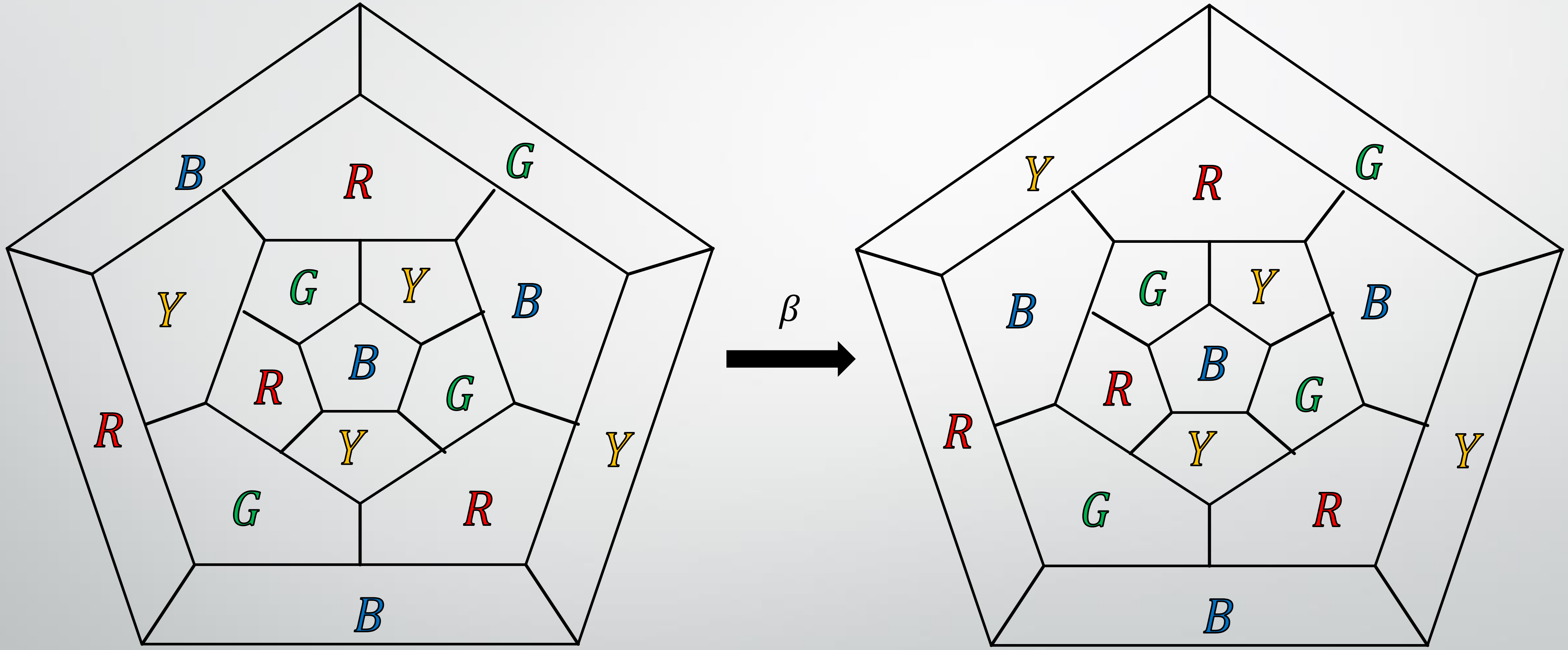

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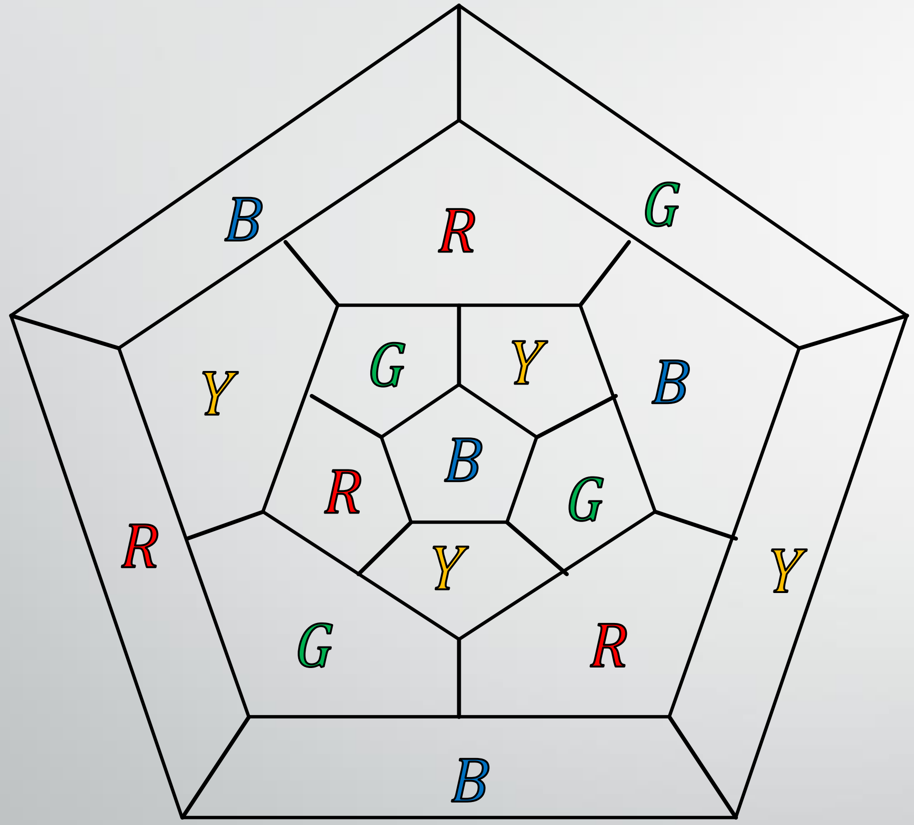

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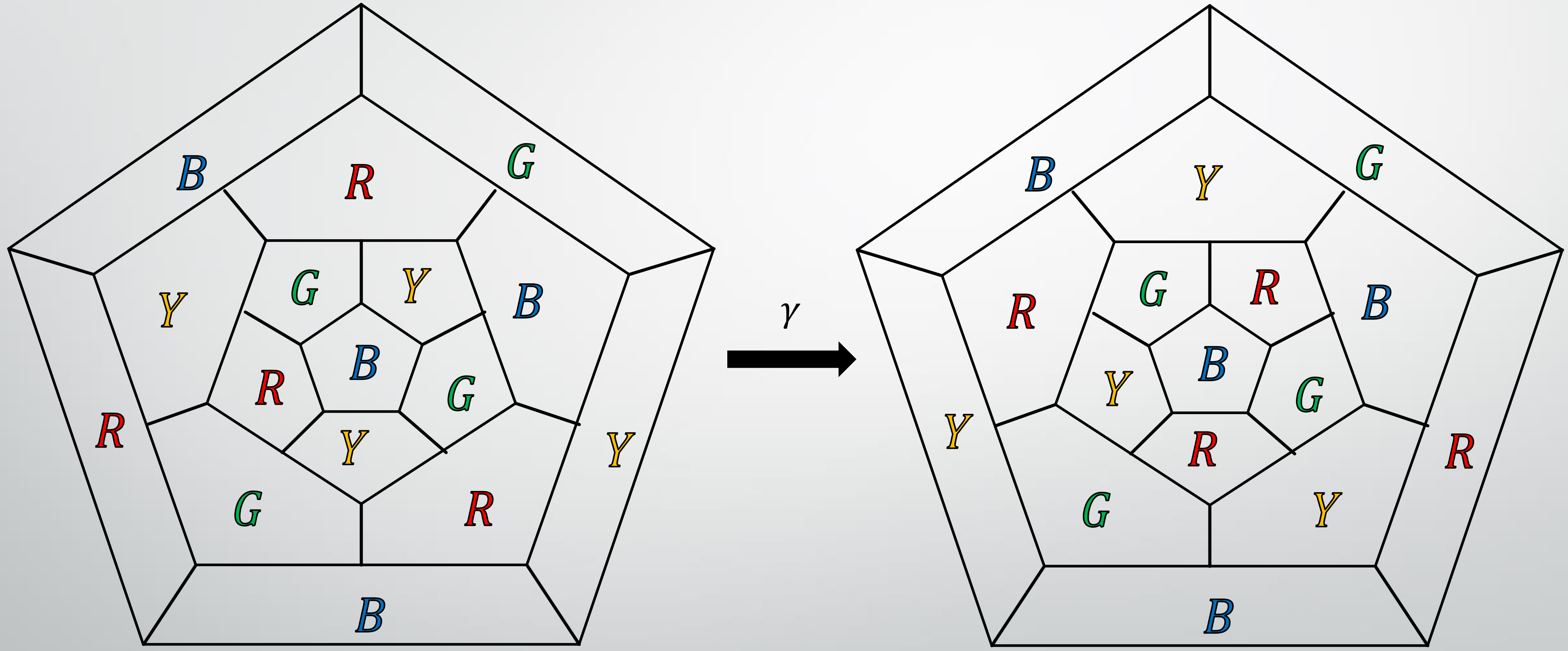

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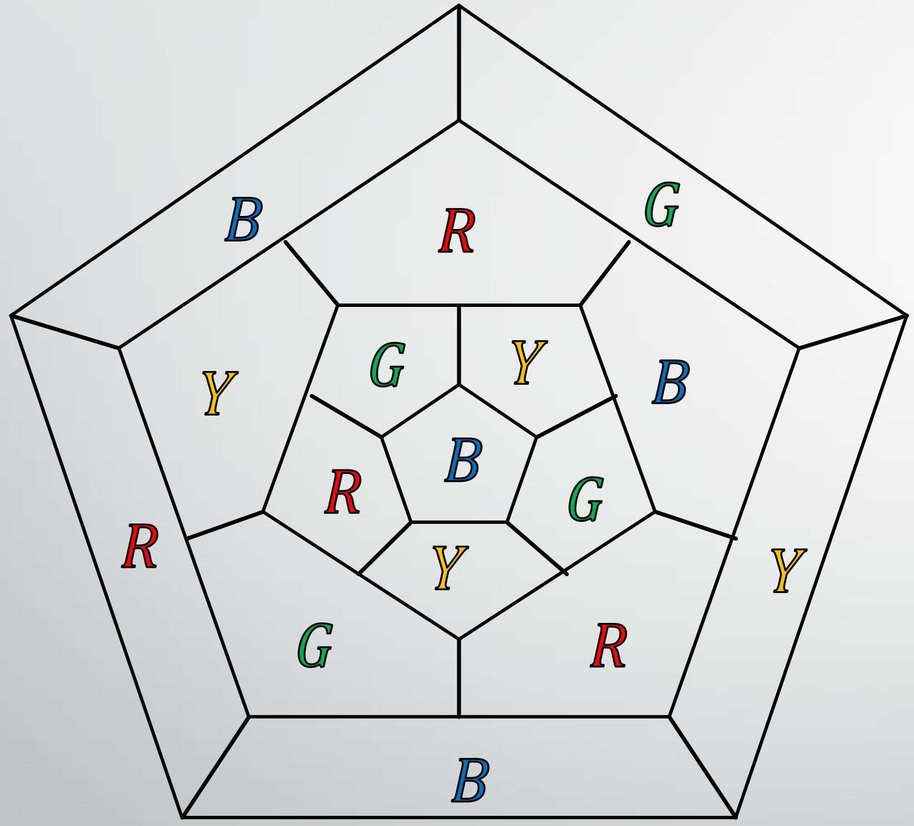

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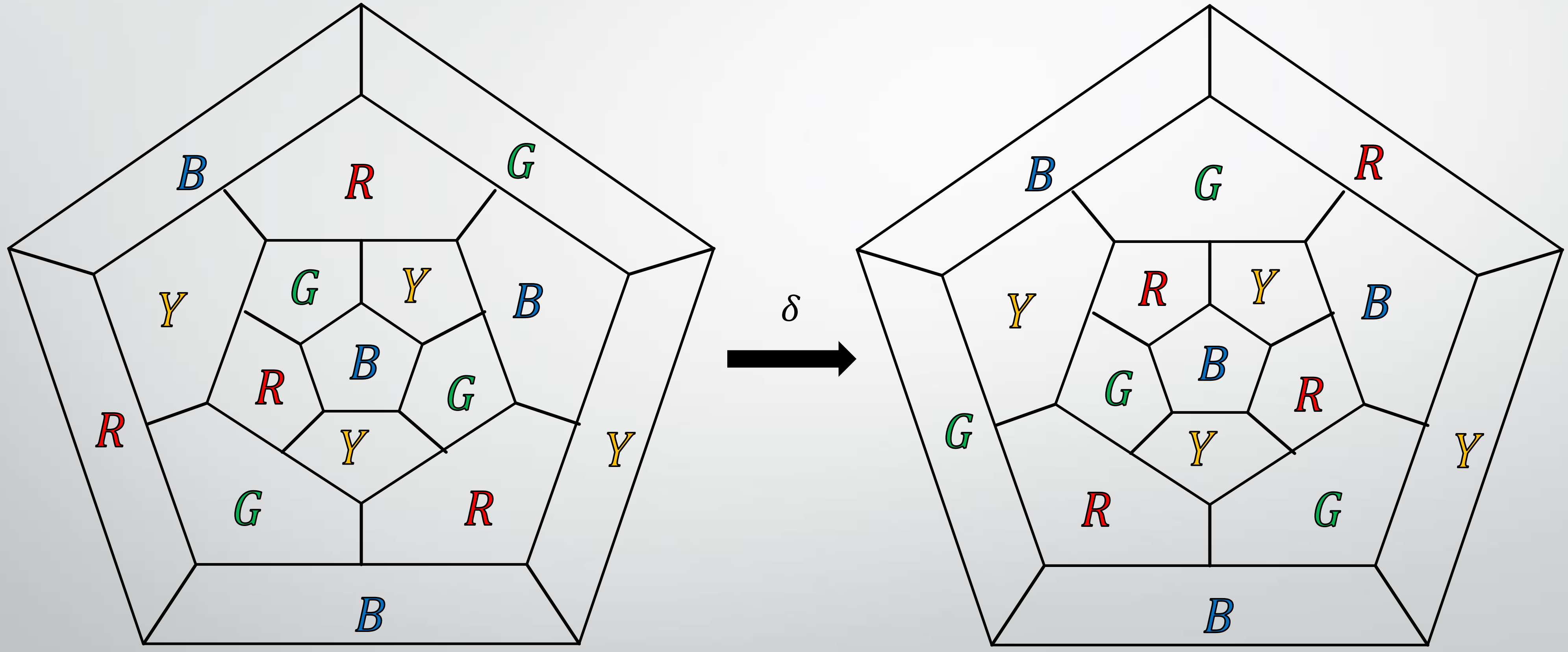

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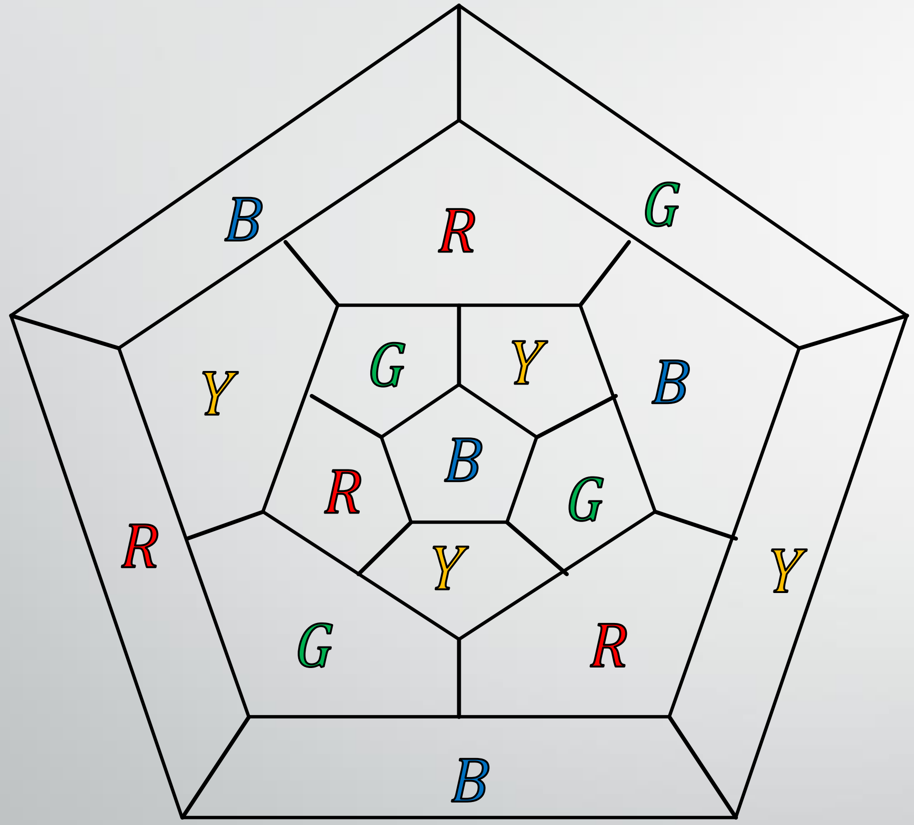

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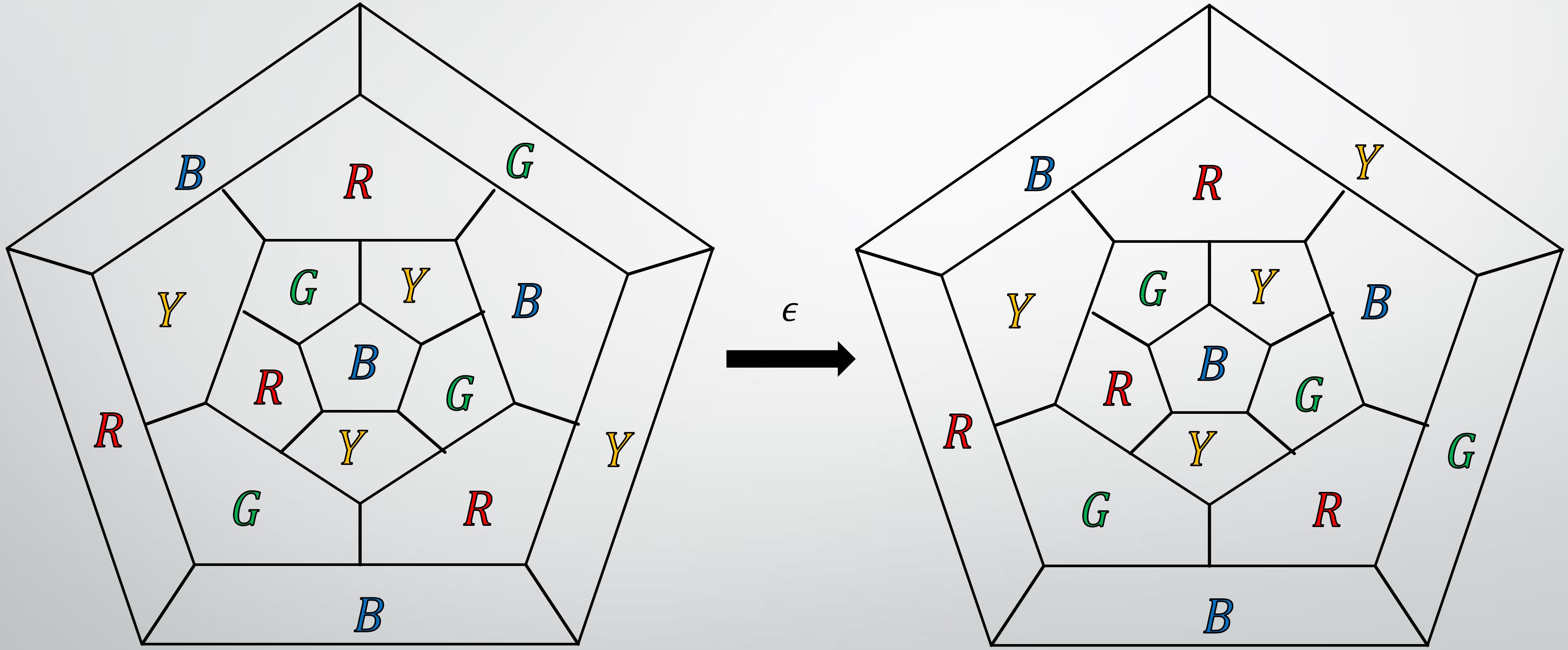

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  - Therefore, these operations would generate a finite group closed under composition.
  - Kittell calls this an *impasse group.*
- This led to a new question:

Question: Can it be shown that the existence of an impasse group closed under all of Kittell's operations results in a contradiction?

The Rotation Method

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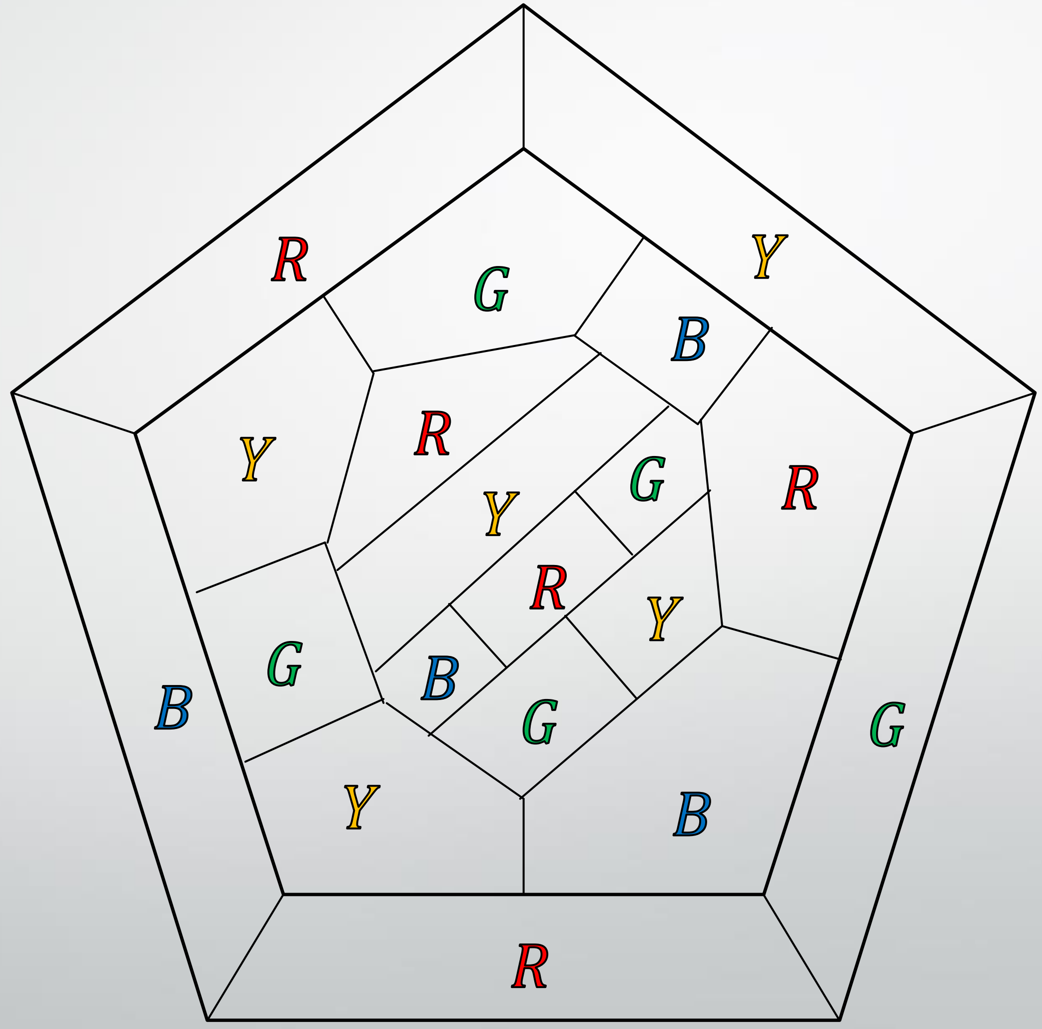
- There are possible variations on the Rotation Method, but we will use this as our definition. (See [9, 10, 11] for more on the Rotation Method.)

An Example



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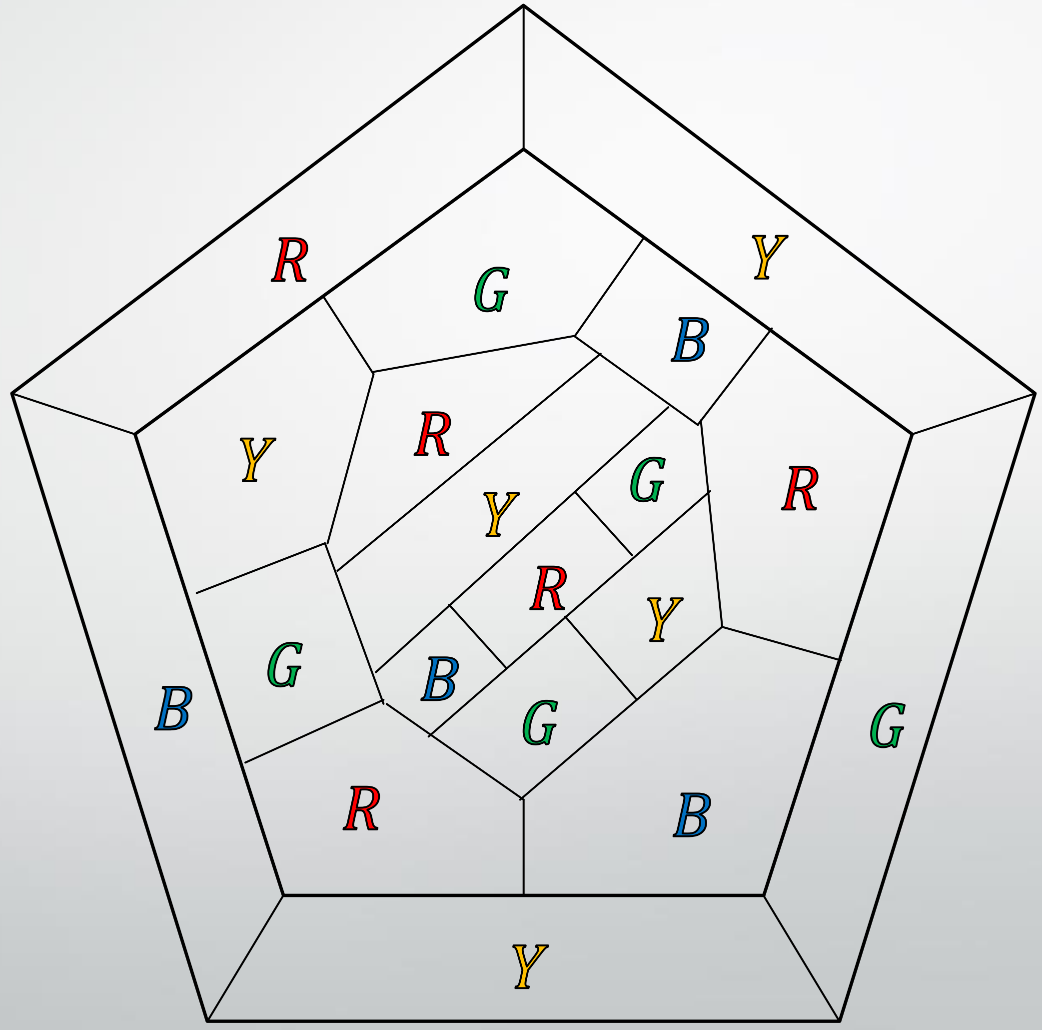

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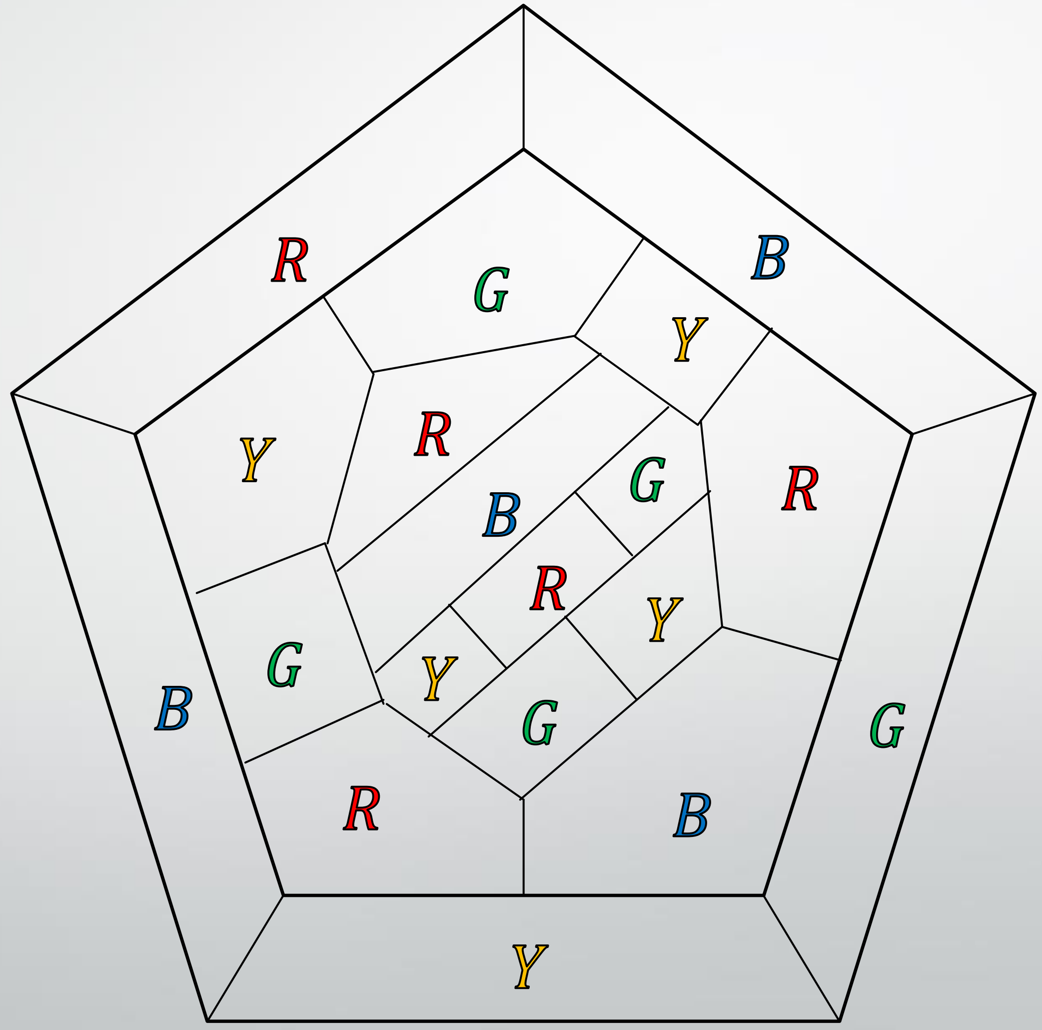

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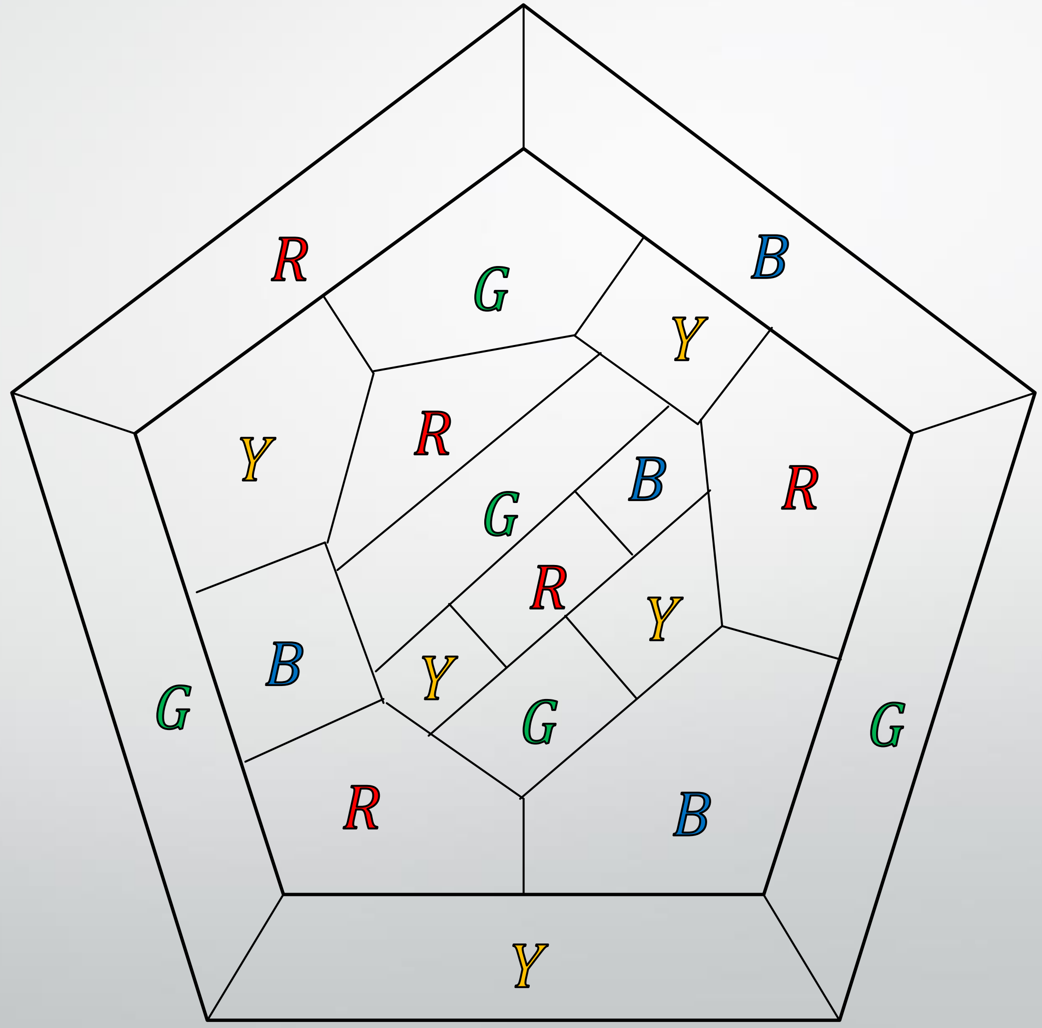

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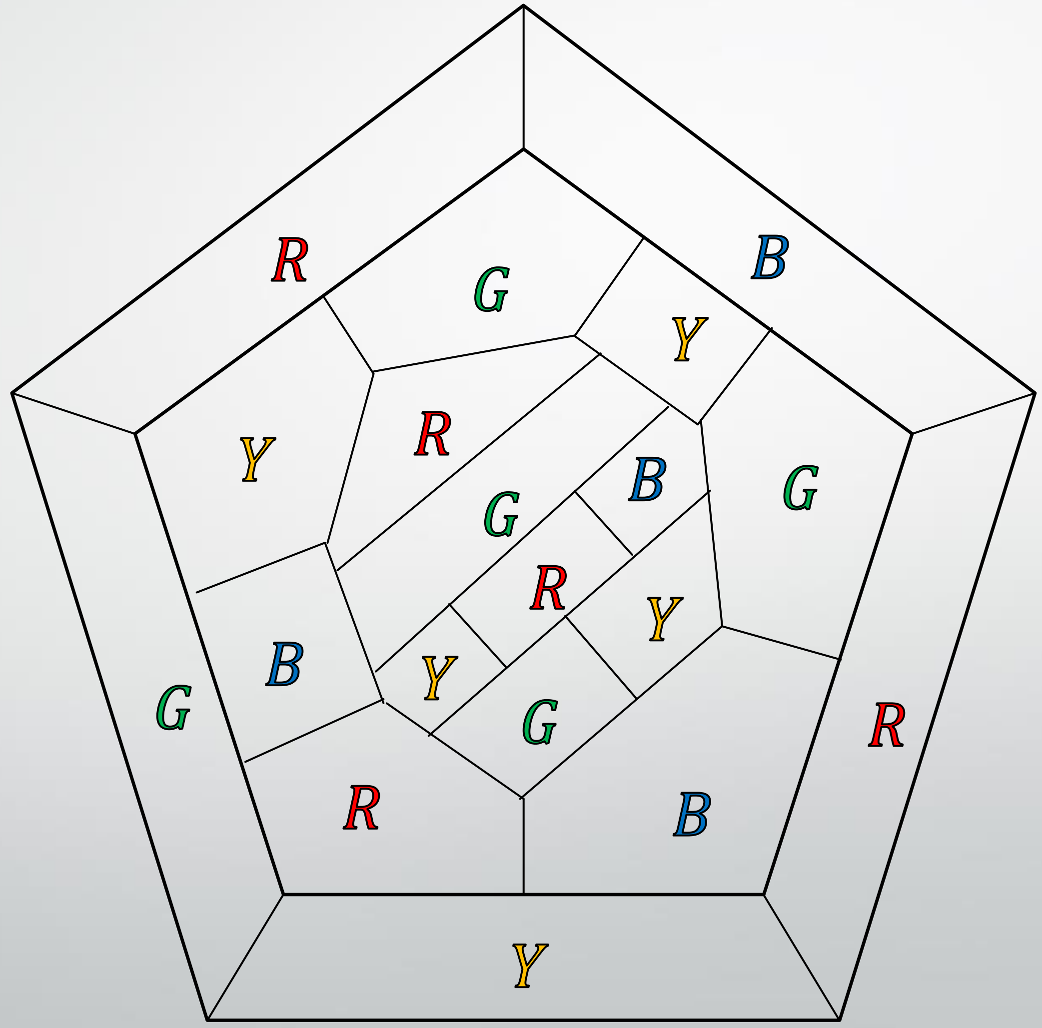

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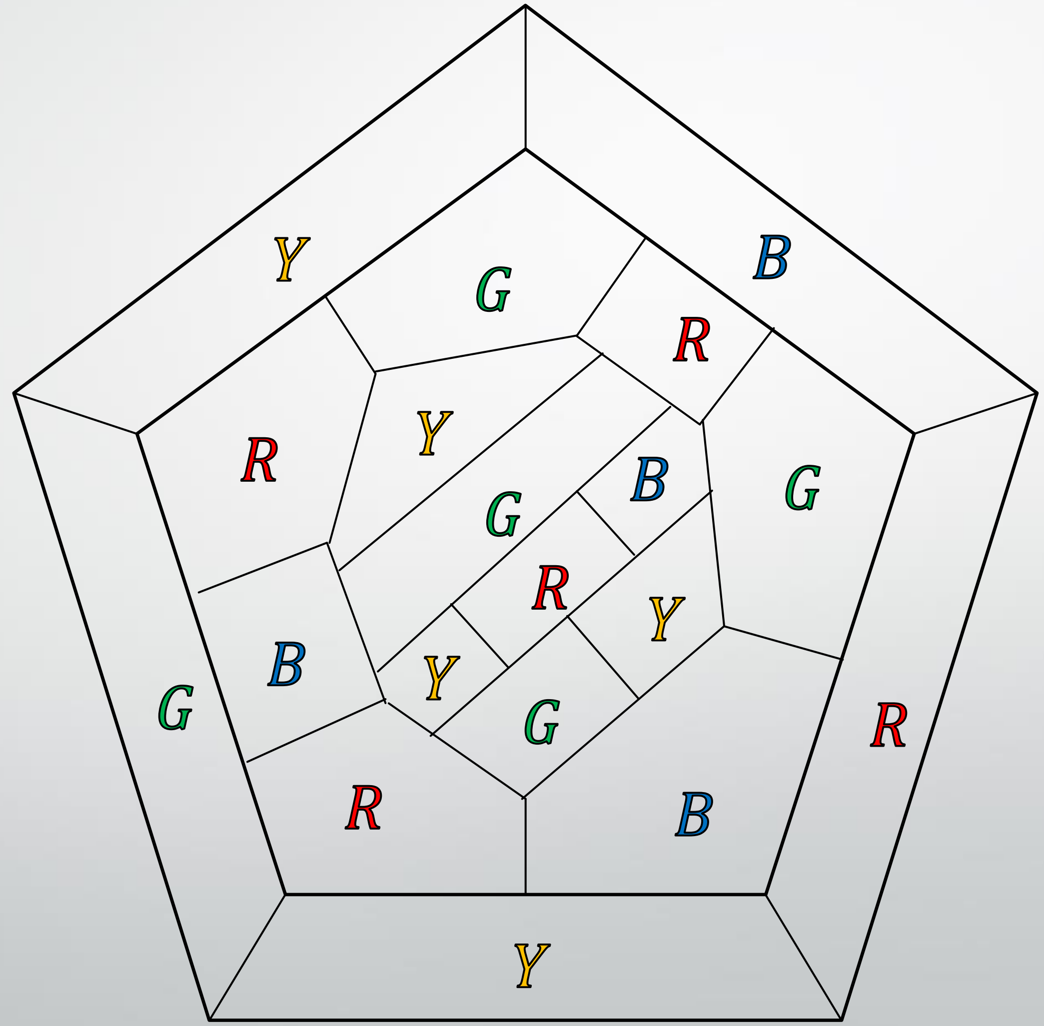

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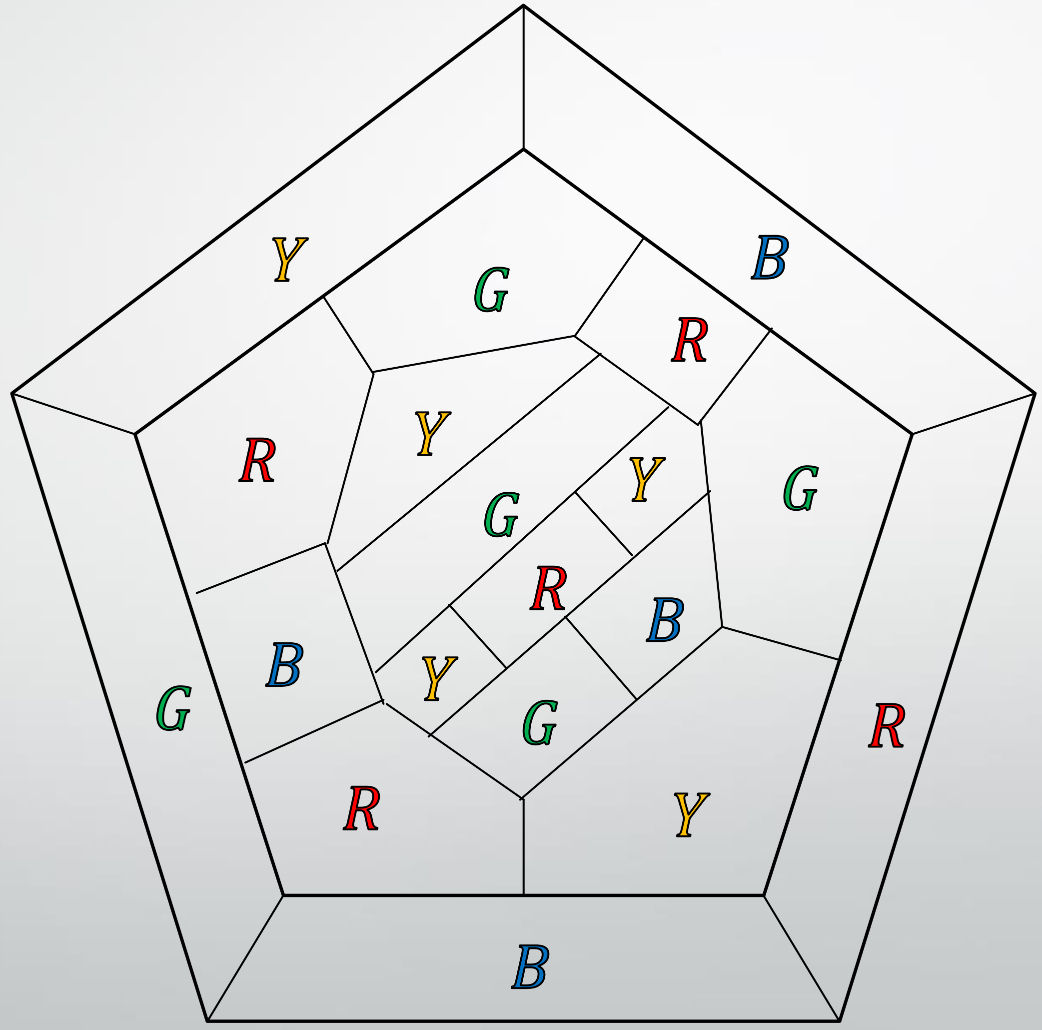

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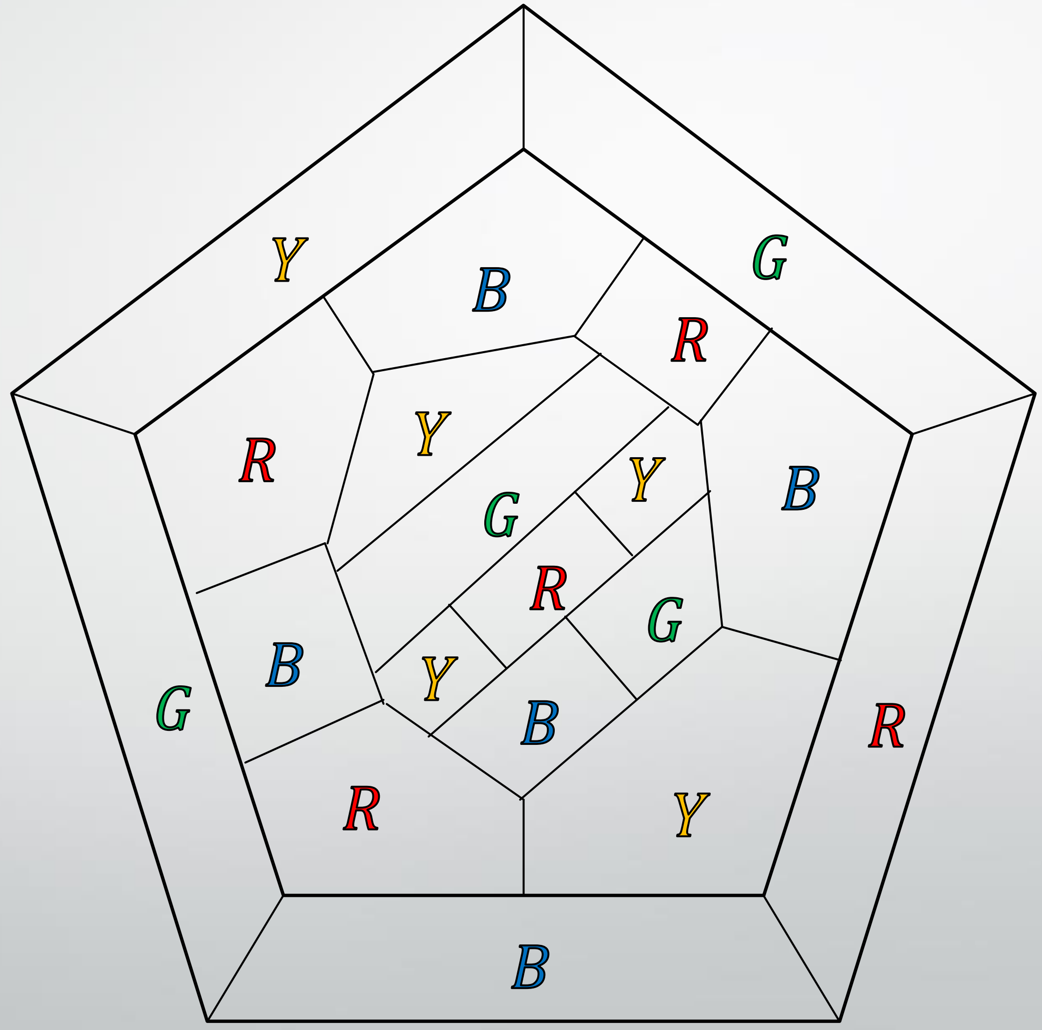

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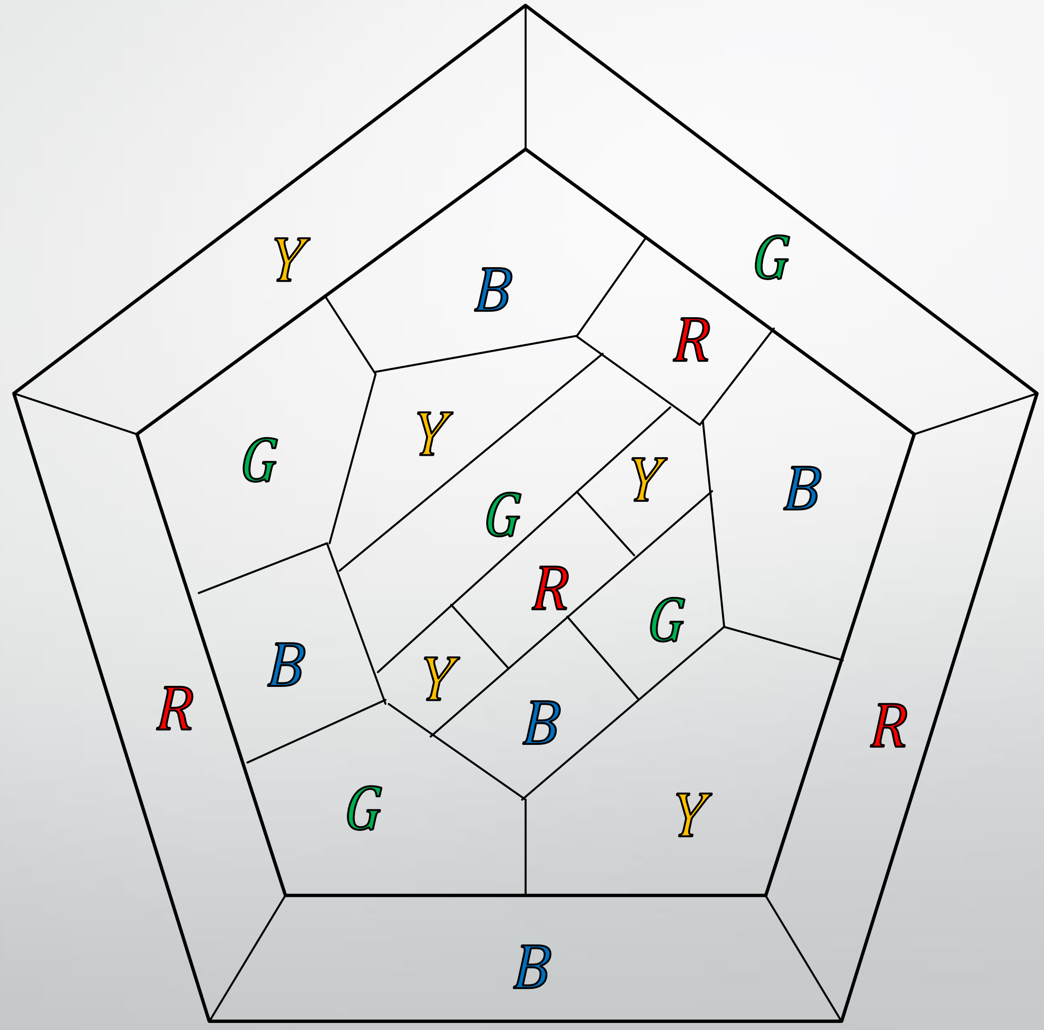

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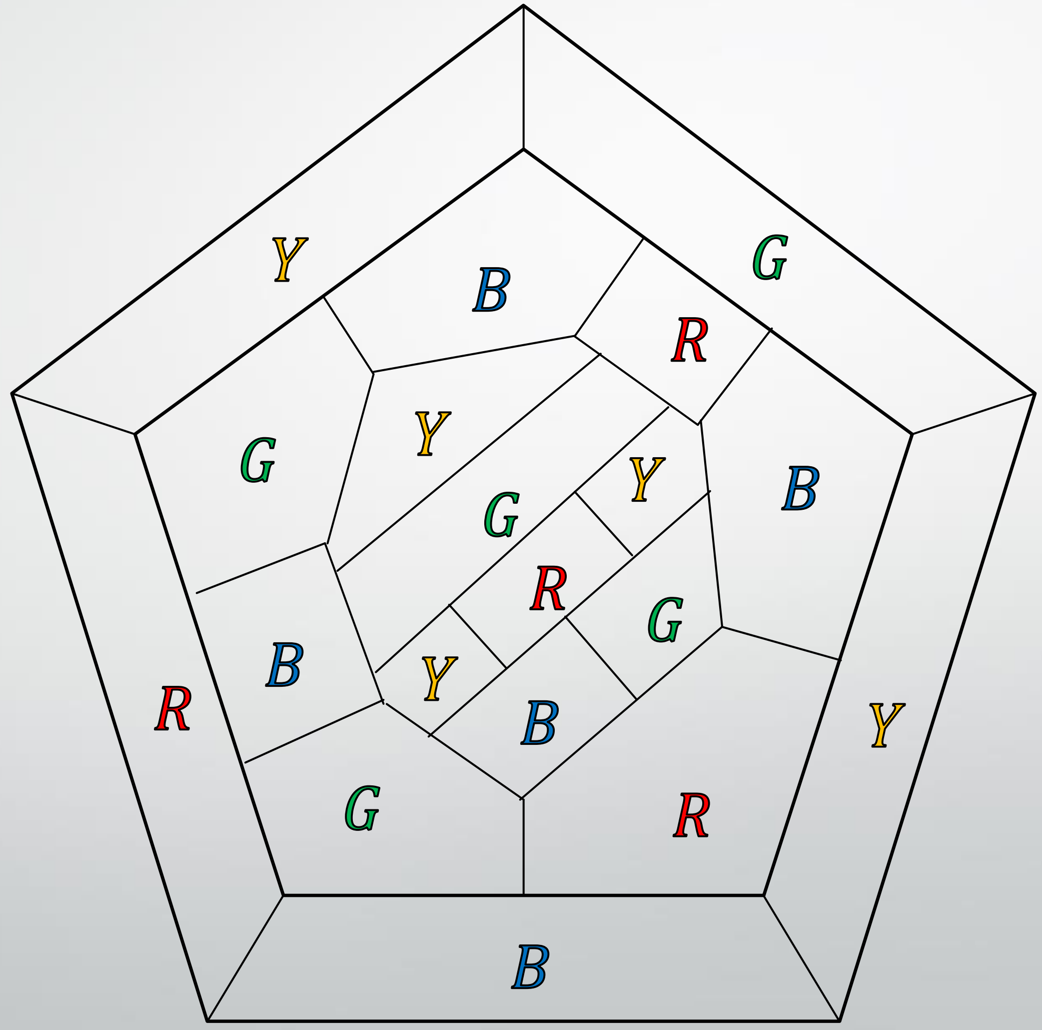

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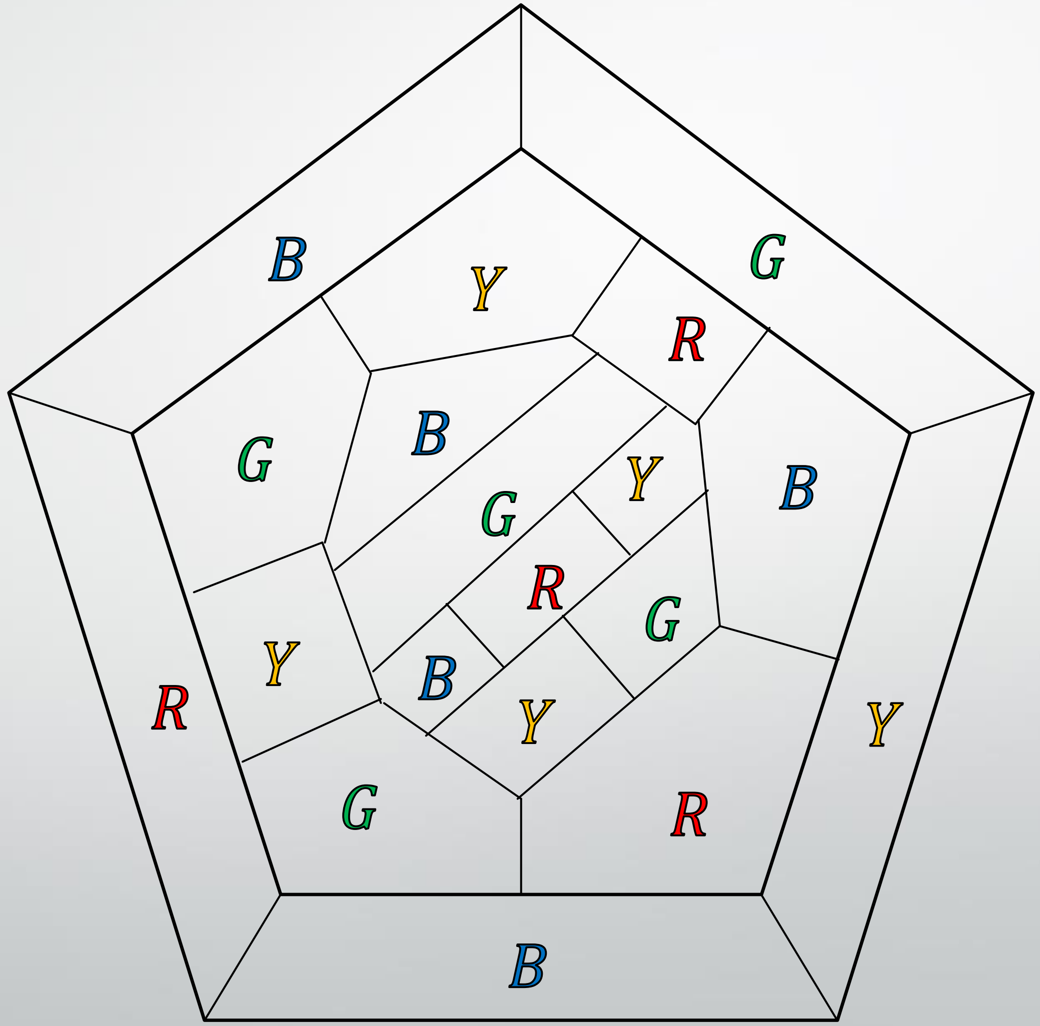

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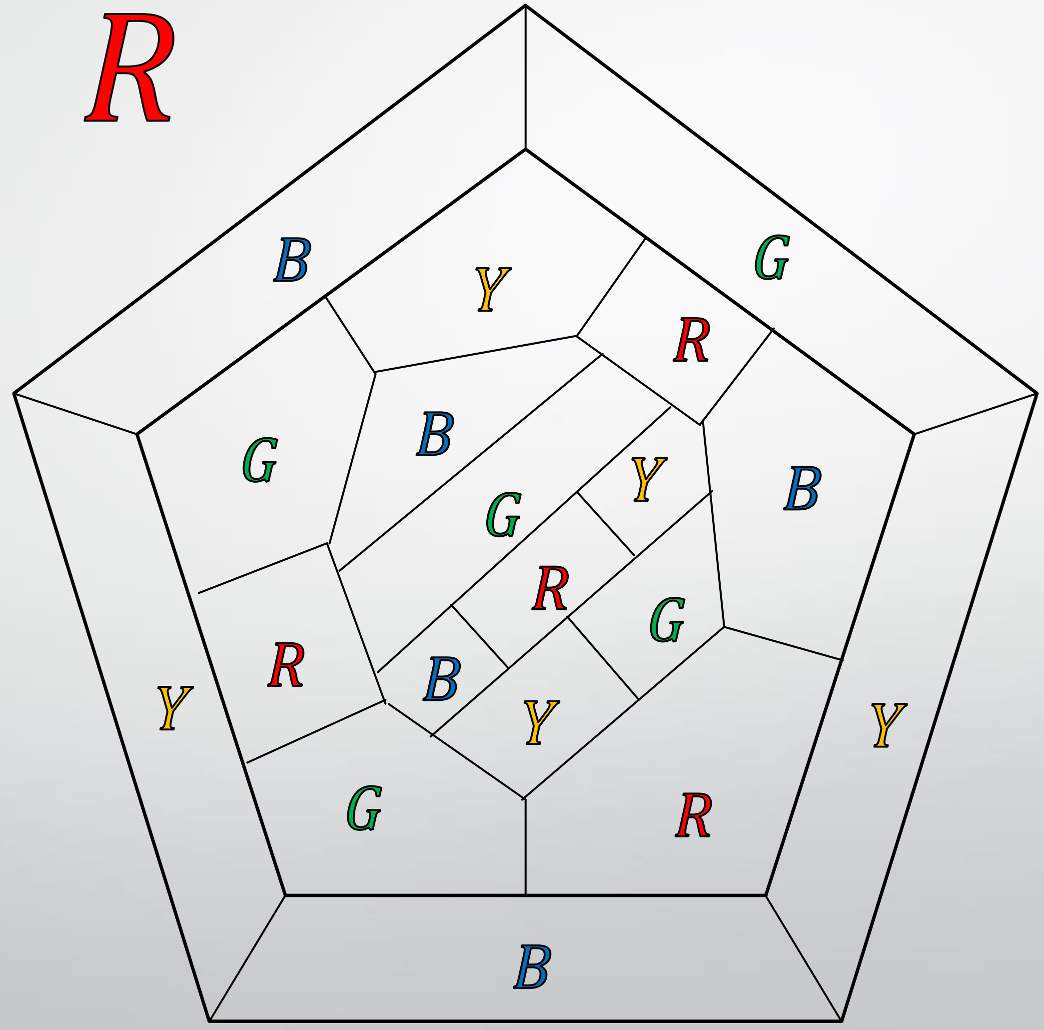

Question: Can this resolve any partial coloring of any map?

The Errera Map

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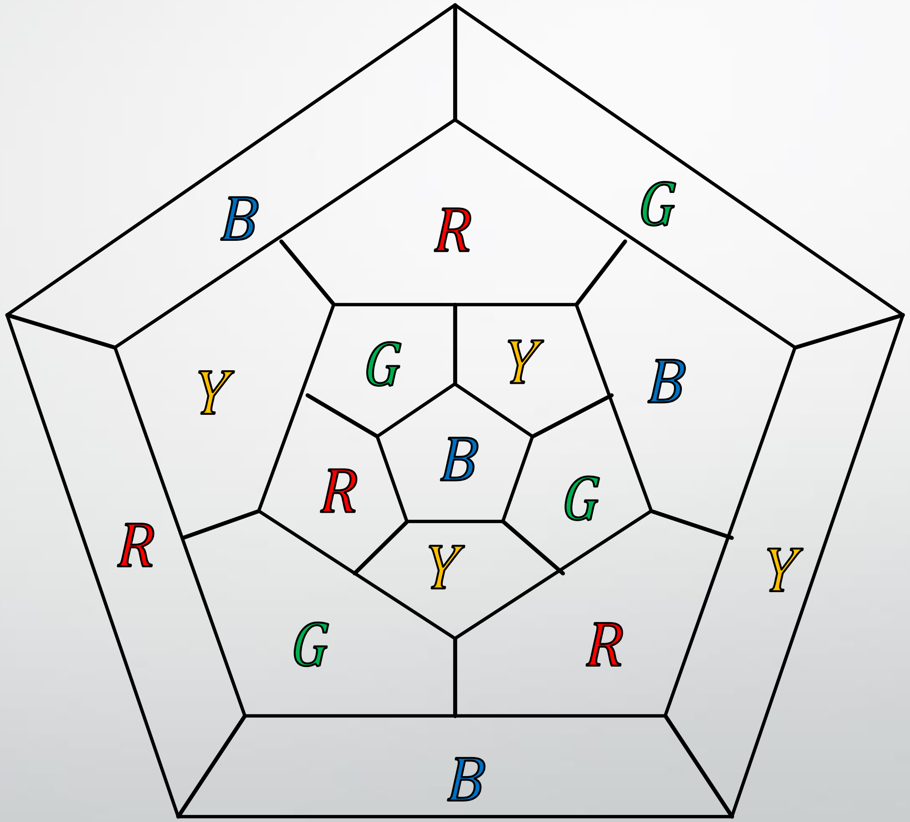
  - In other words, α generates a group of order 20.
- After just 4 applications of α, we obtain a rotation of the original coloring.

The Errera Map



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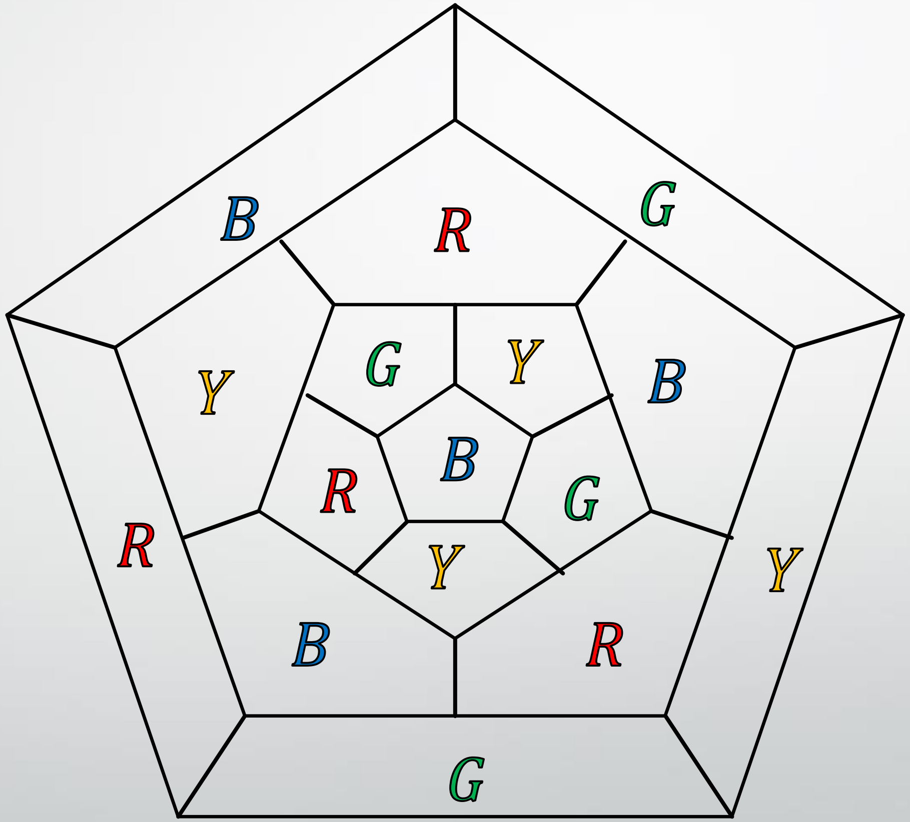

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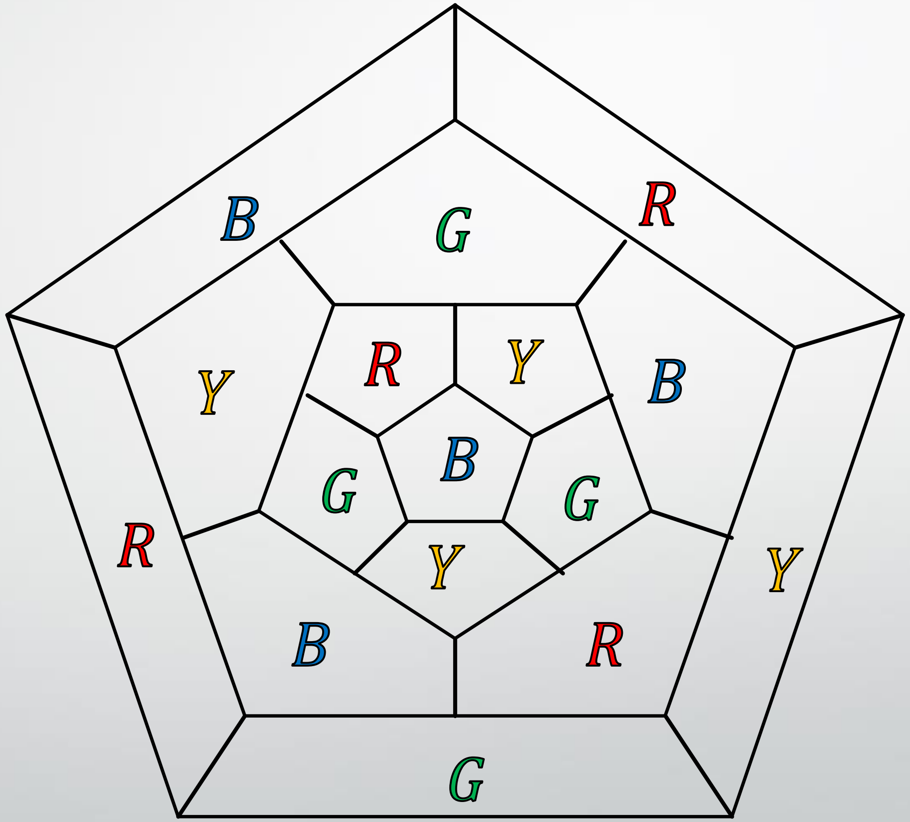

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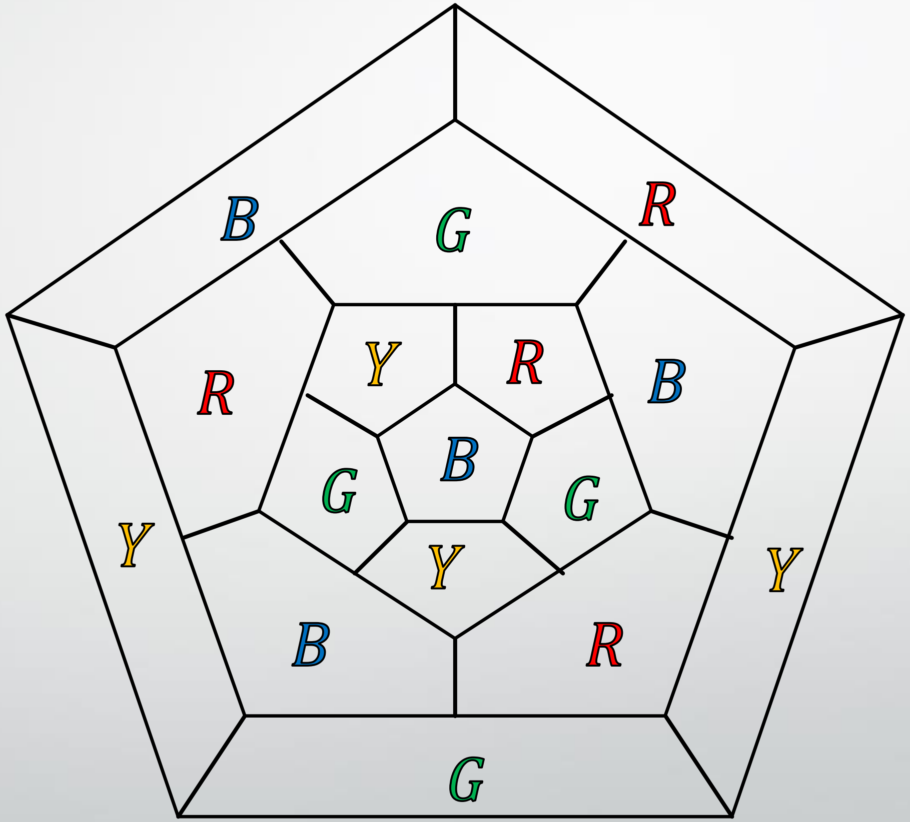

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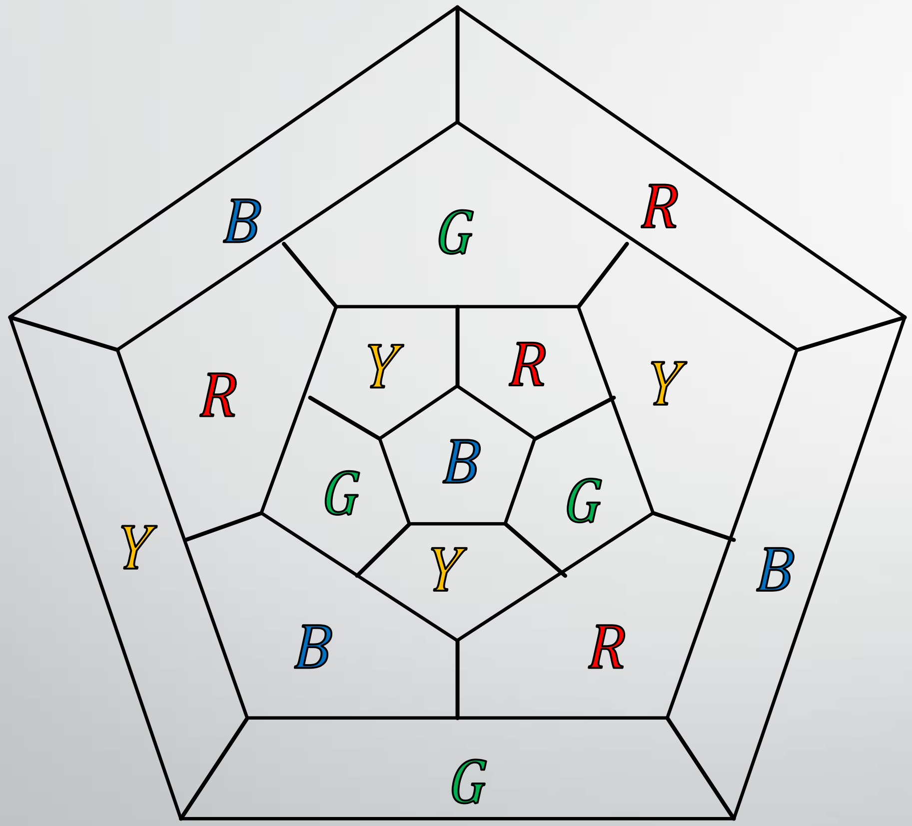

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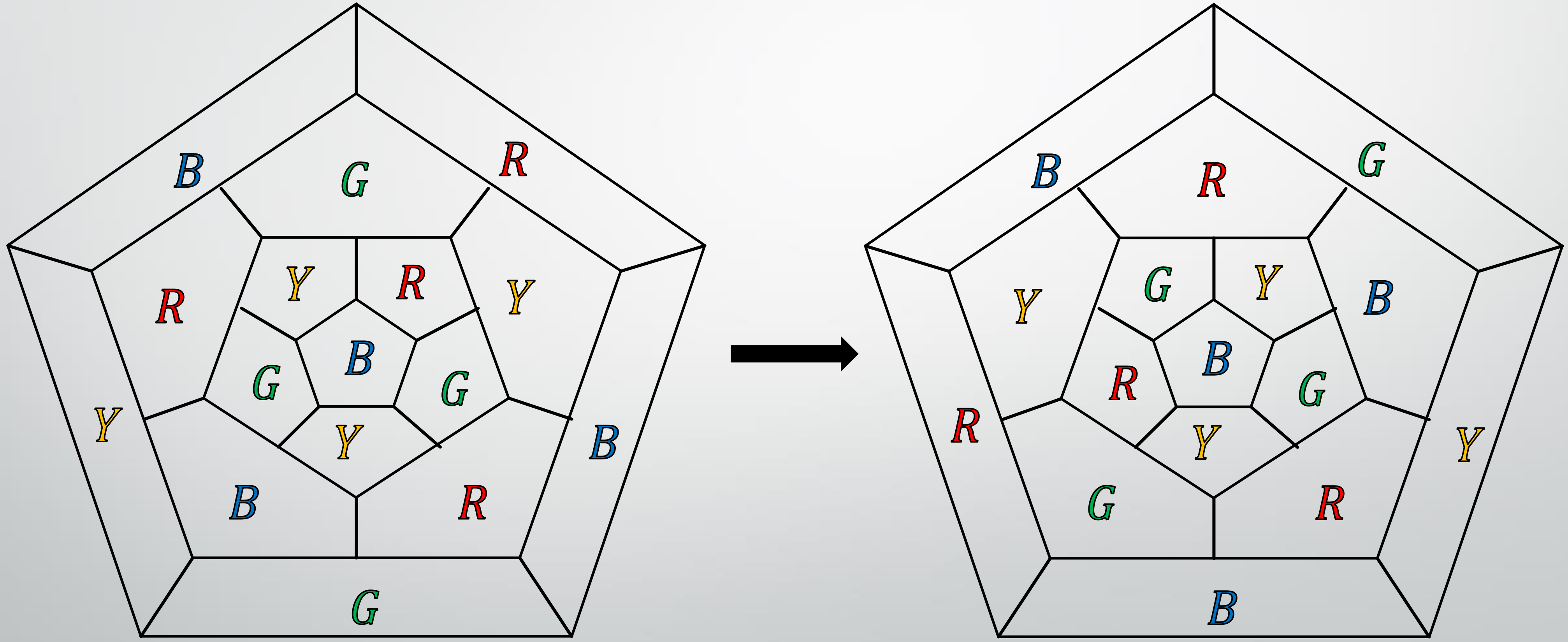

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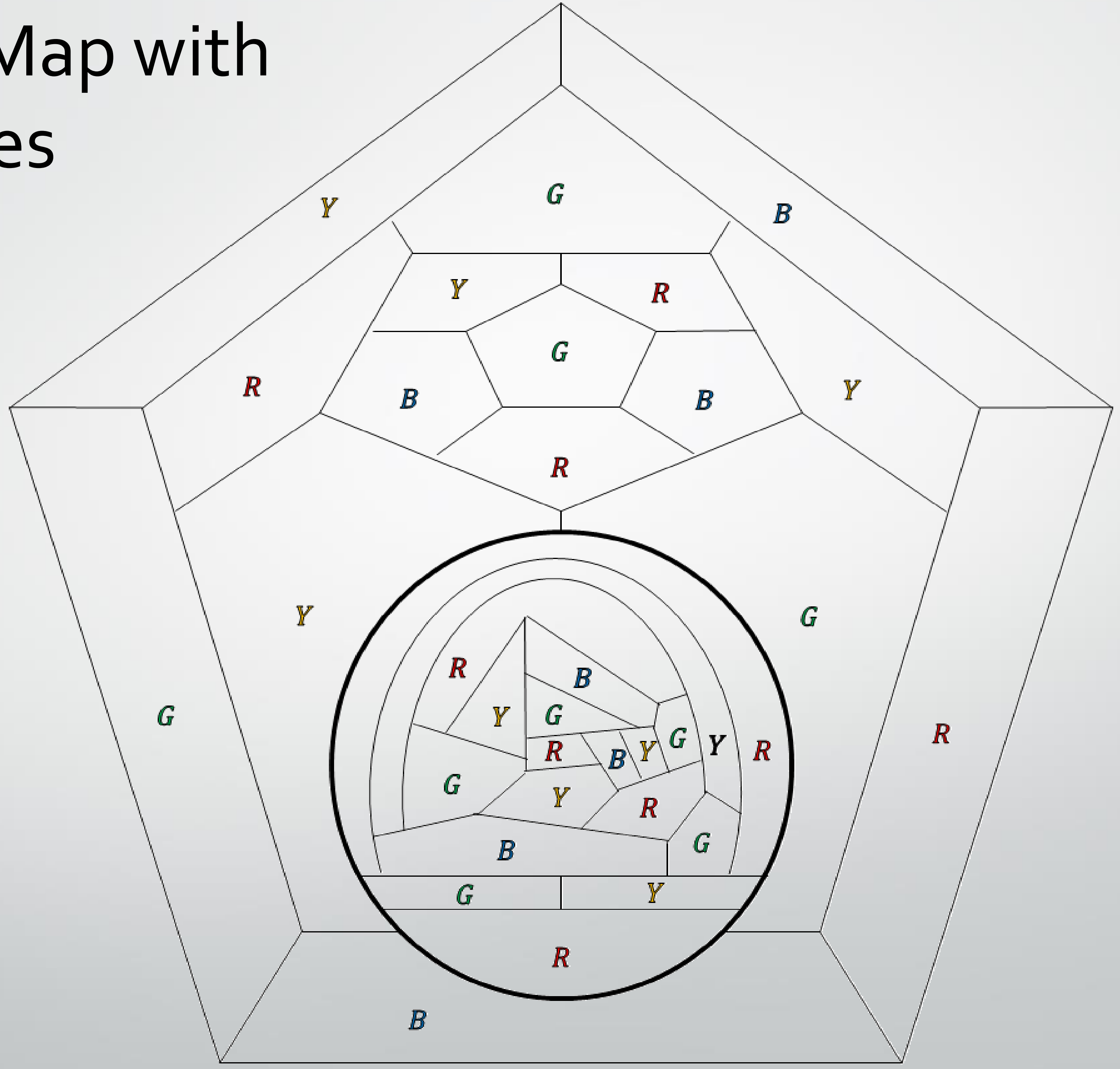
  - In the dual graph, this is a graph containing the Errera map as an induced subgraph.
- In the next slide is an example of an Errera Map with Holes, where $|\langle\alpha\rangle| = 60$.

An Errera Map with Holes



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- To this end, we would like to study exactly which colorings of the Errera map lead to these problems.

Question: How many colorings of the Errera map result in this cyclic pattern?

Answer: 4

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- Or 480, depending on how you count.

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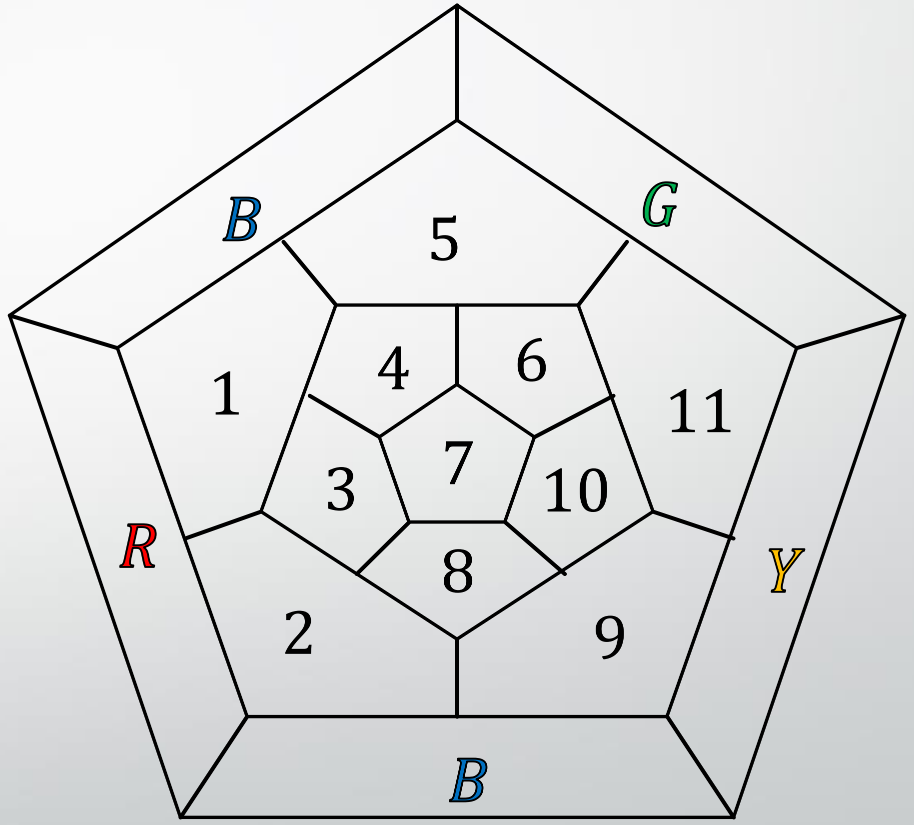
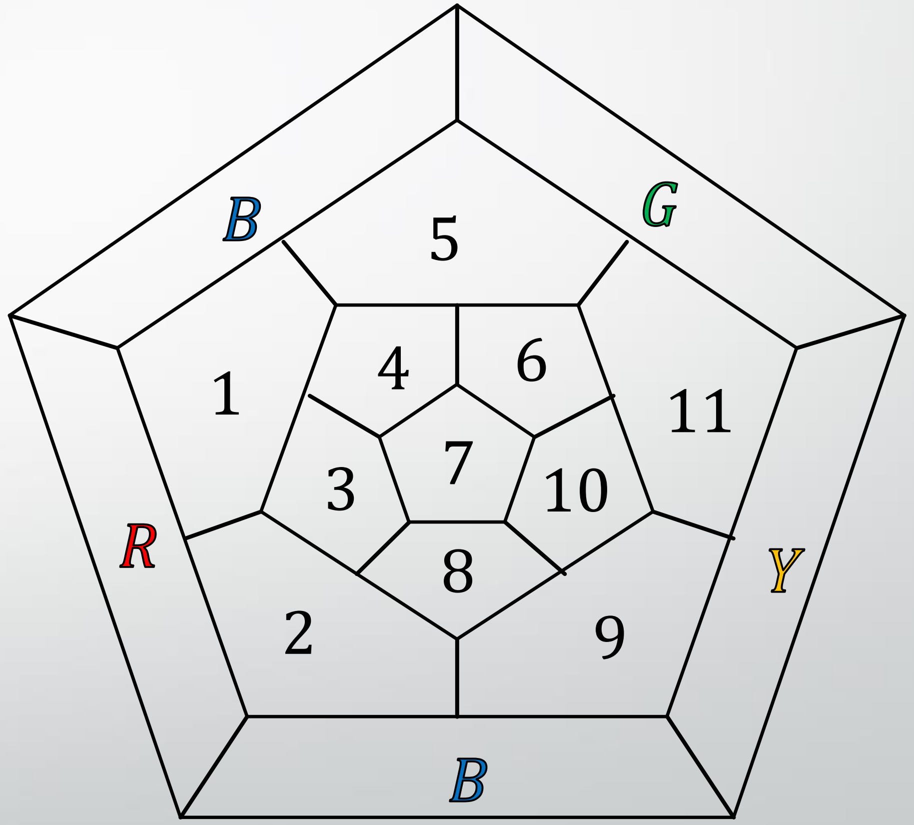
  - We proceed by (a reasonable number of) cases.
  - We will do one such case here.

Determining the Colorings



Determining the Colorings

- We start by coloring the boundary regions arbitrarily, using all 4 colors.



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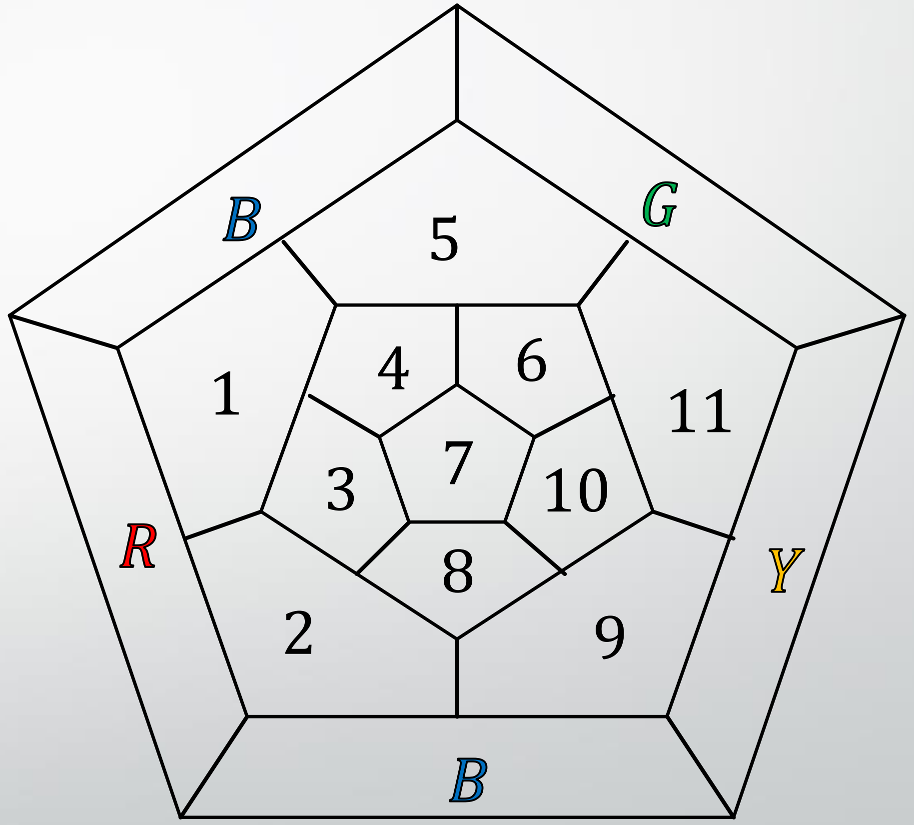

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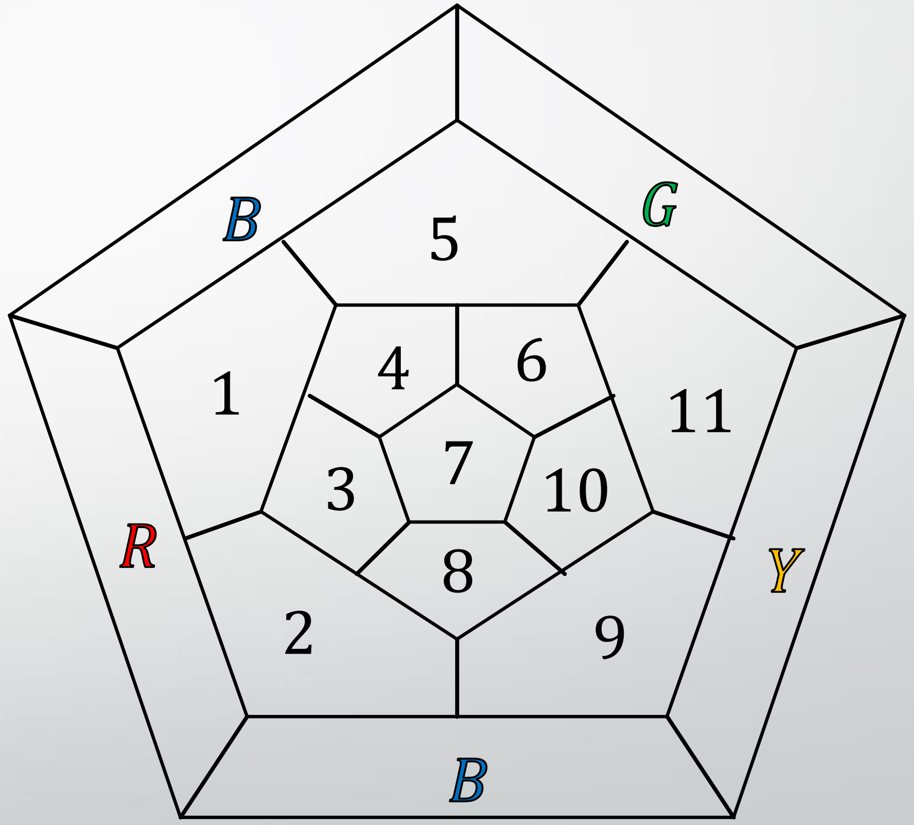

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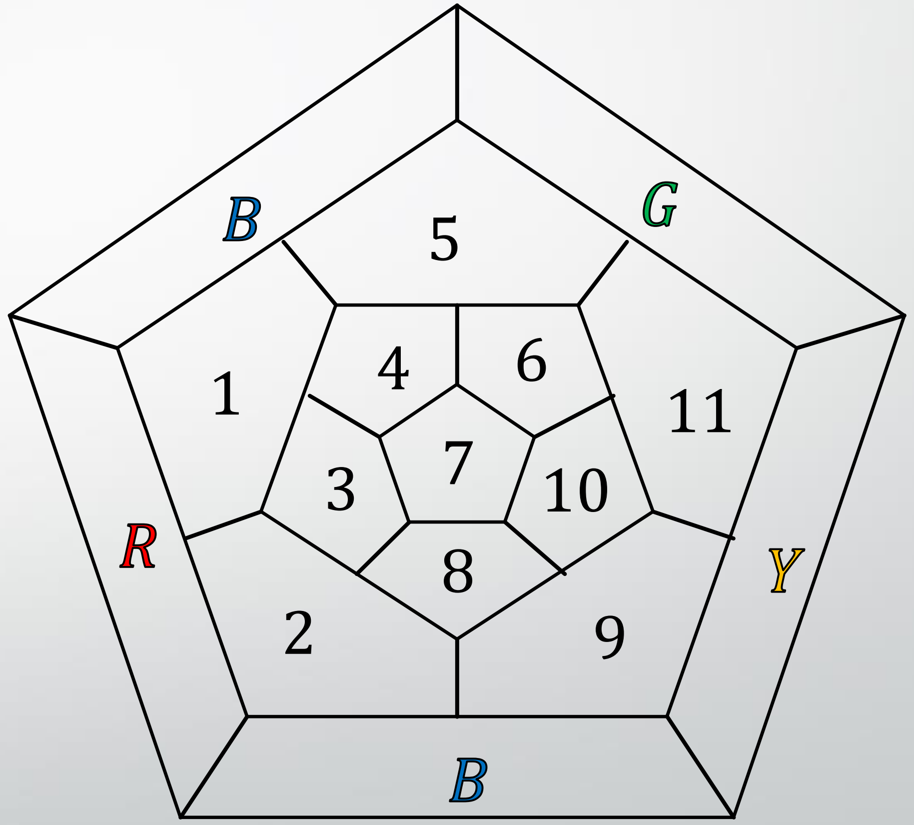

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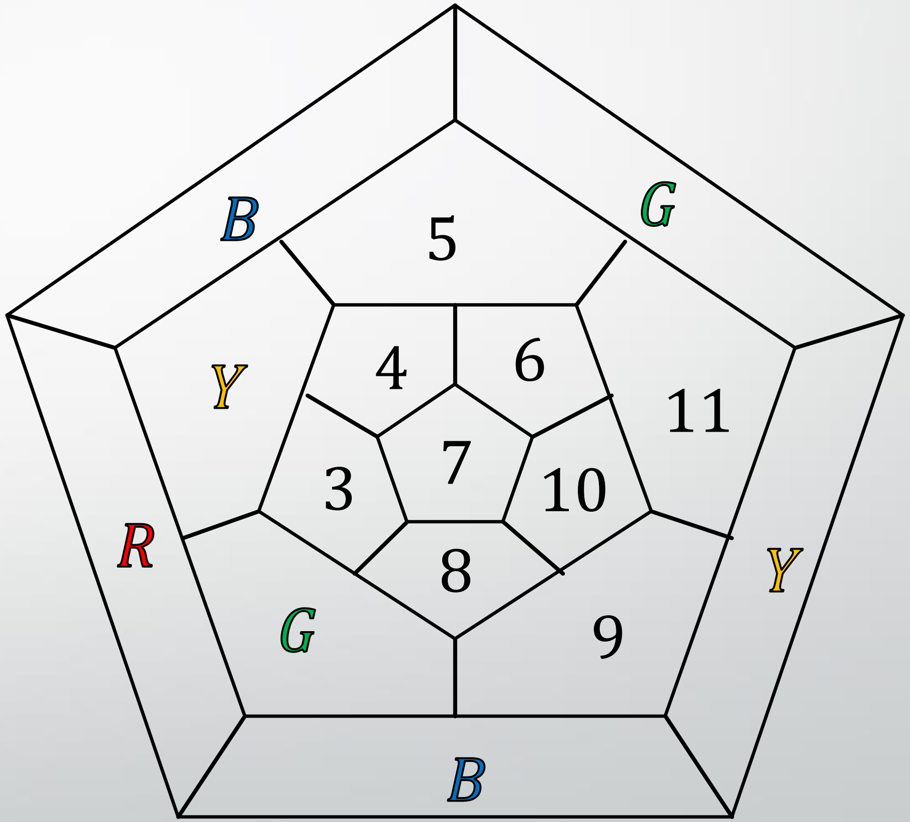

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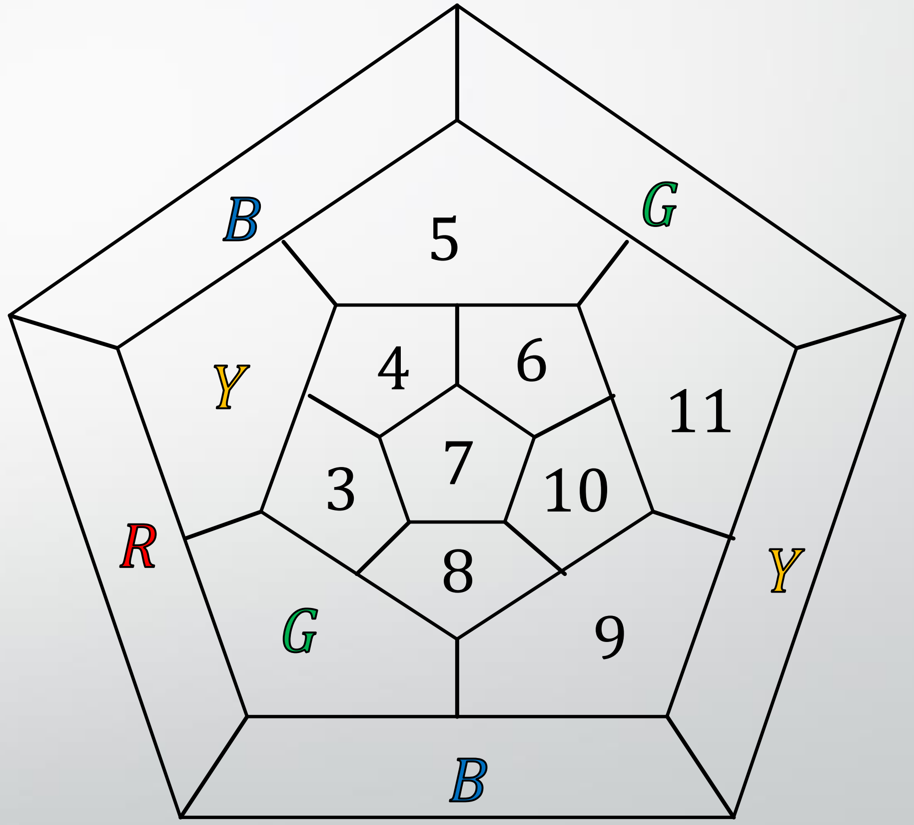

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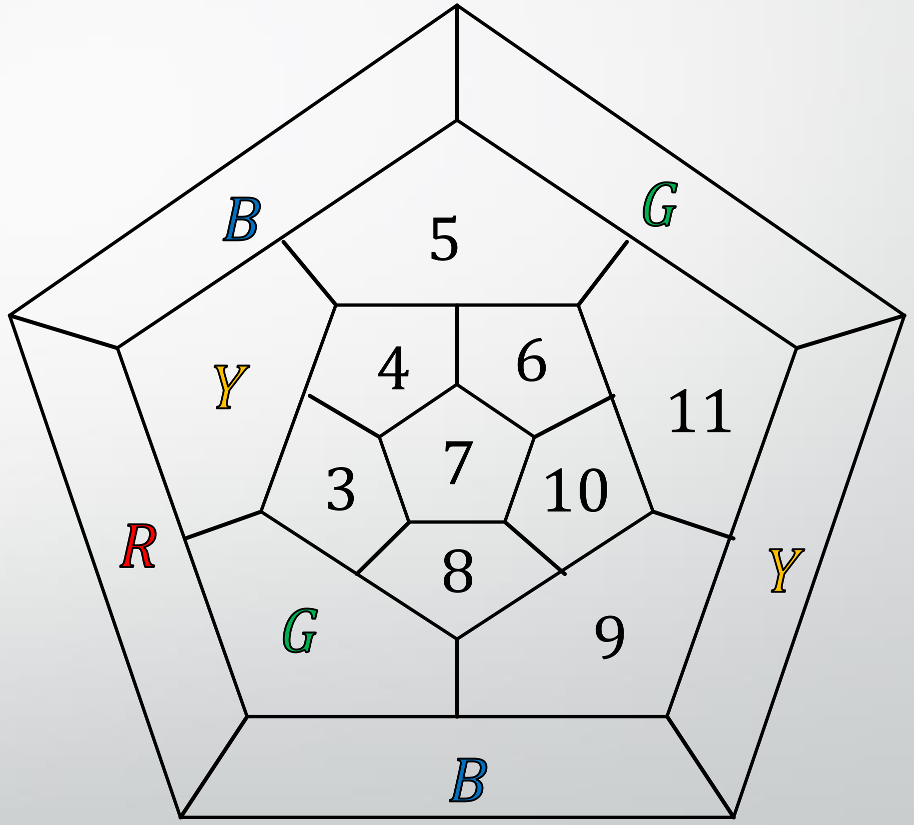
- Since this is an impasse coloring, it must have an RY circuit. We will build that RY circuit.



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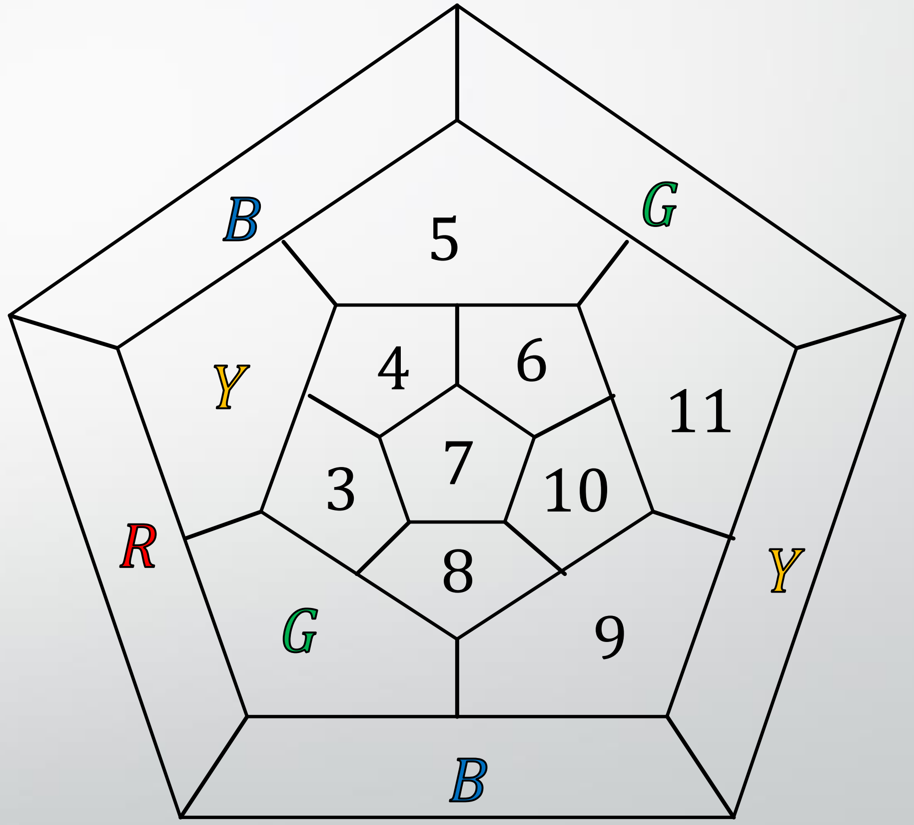

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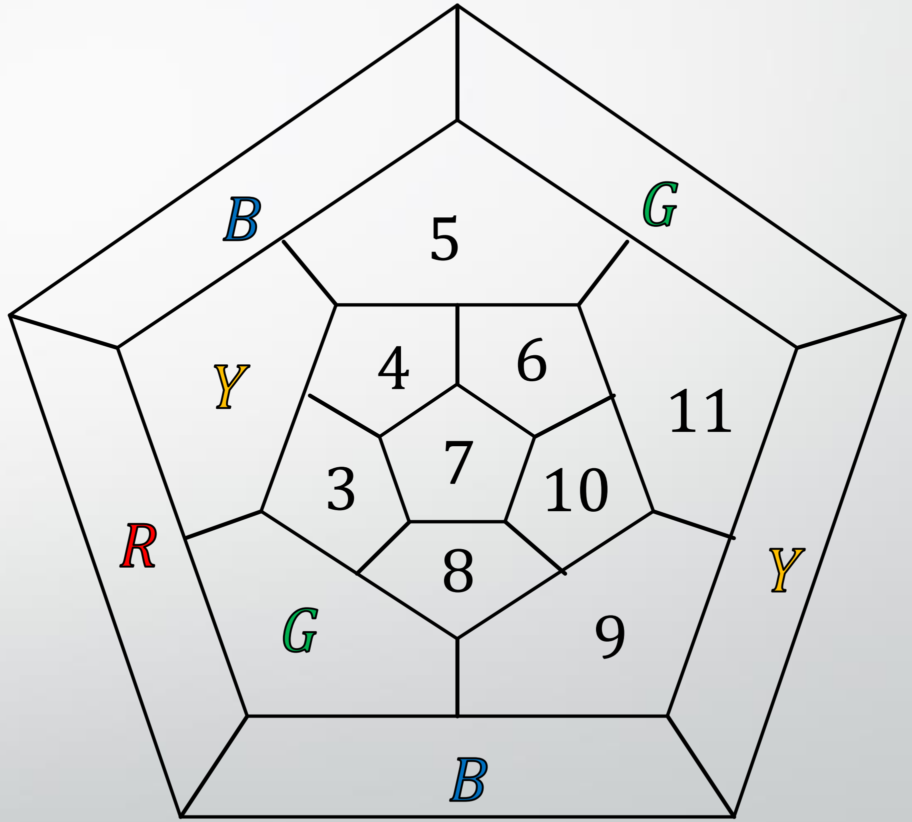

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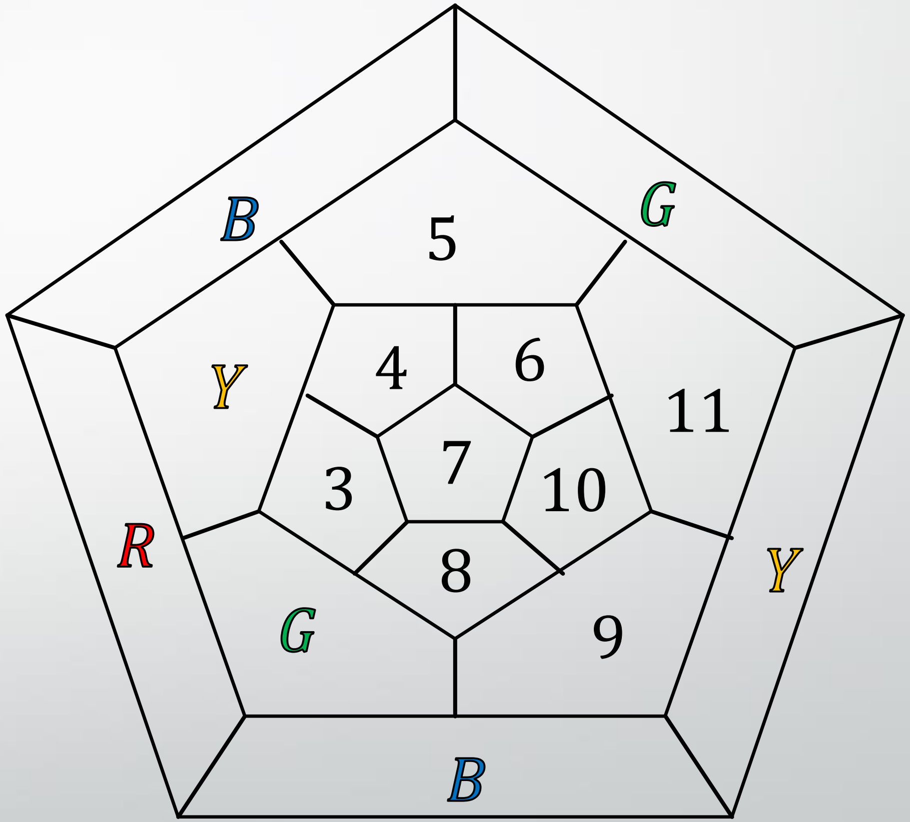

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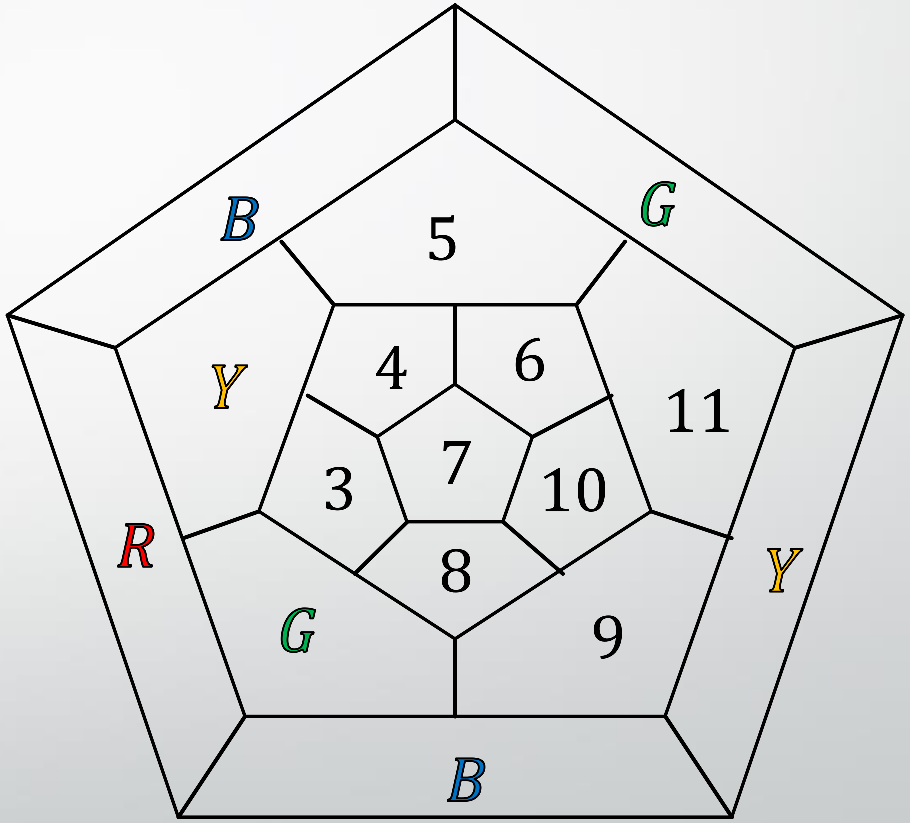

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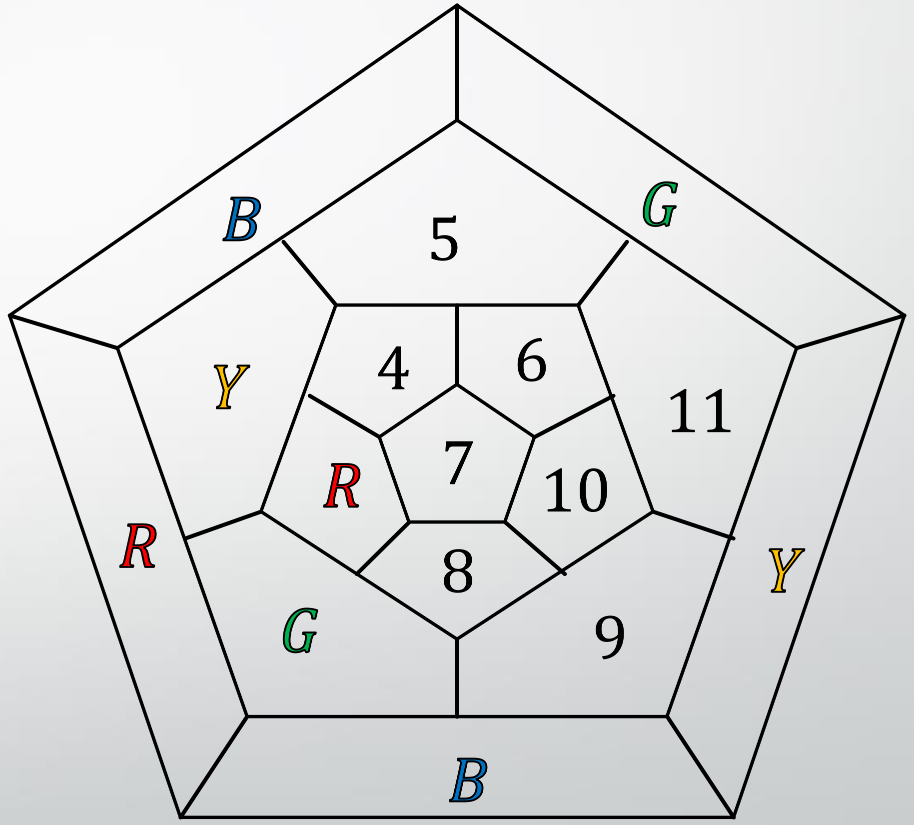

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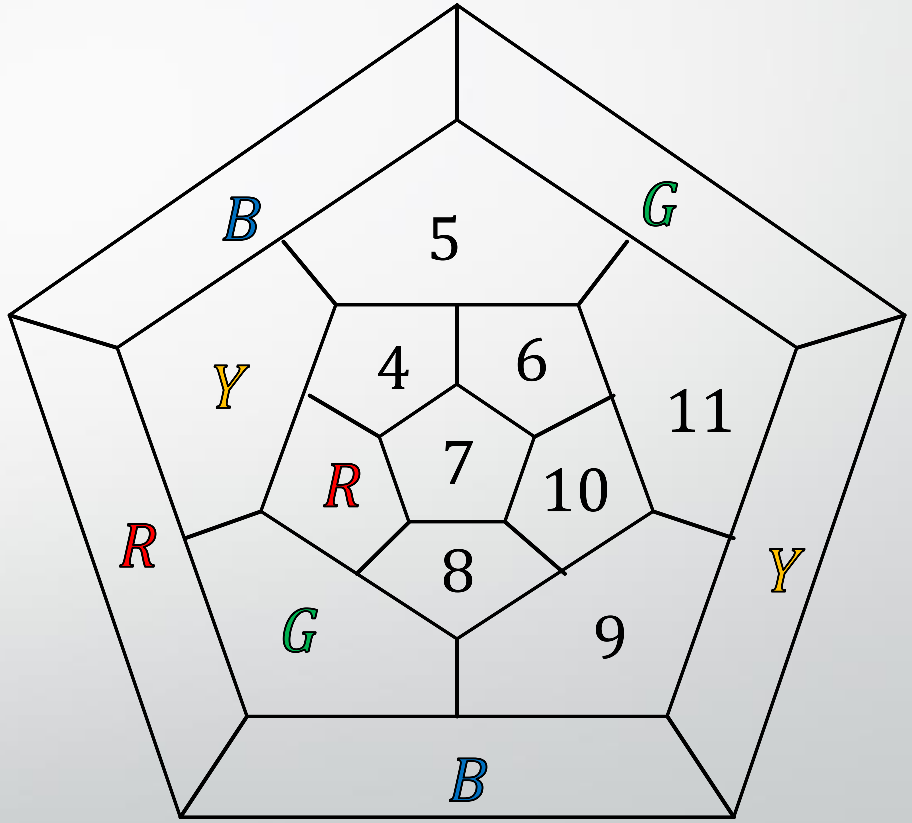

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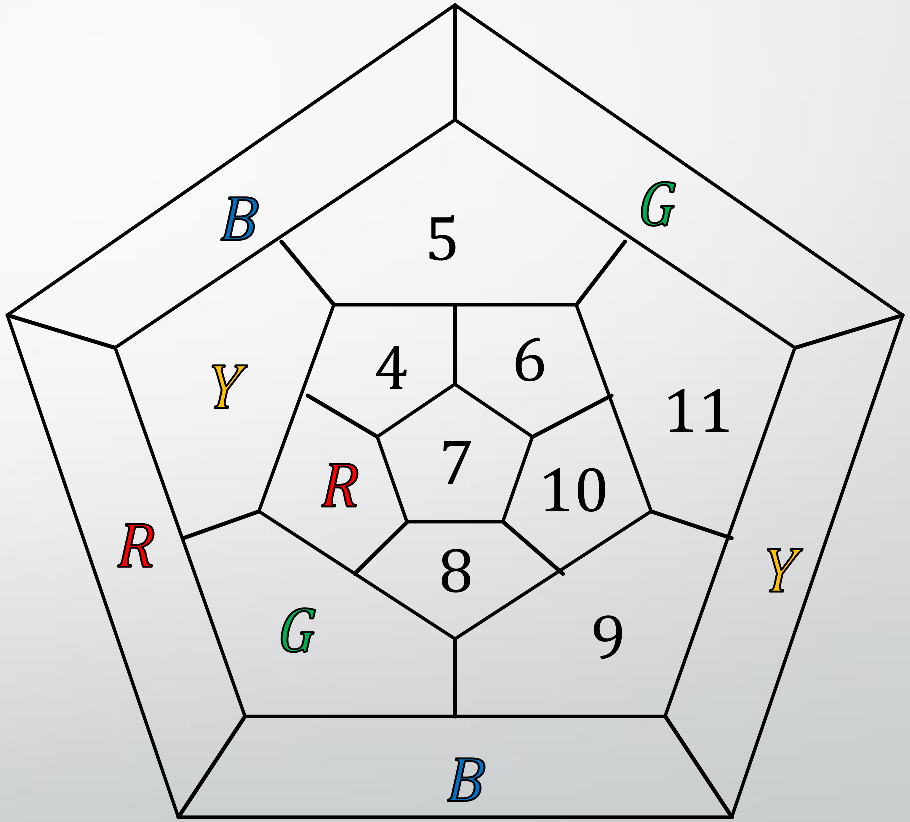
- The next region of our circuit must be 7 or 8. Suppose it were 7.



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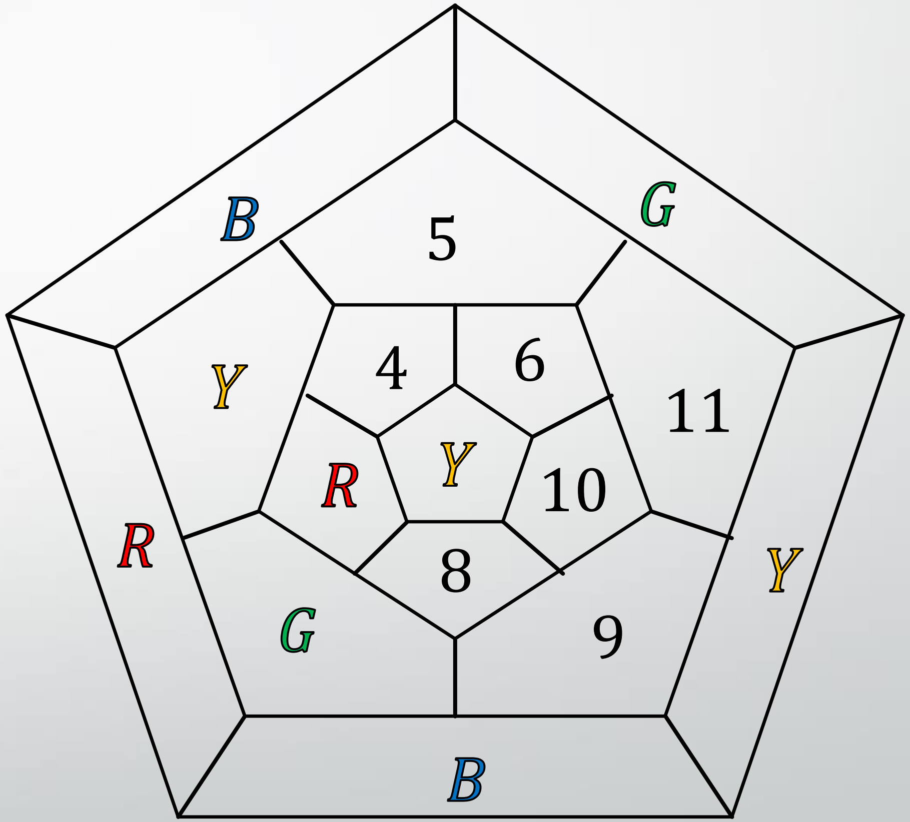

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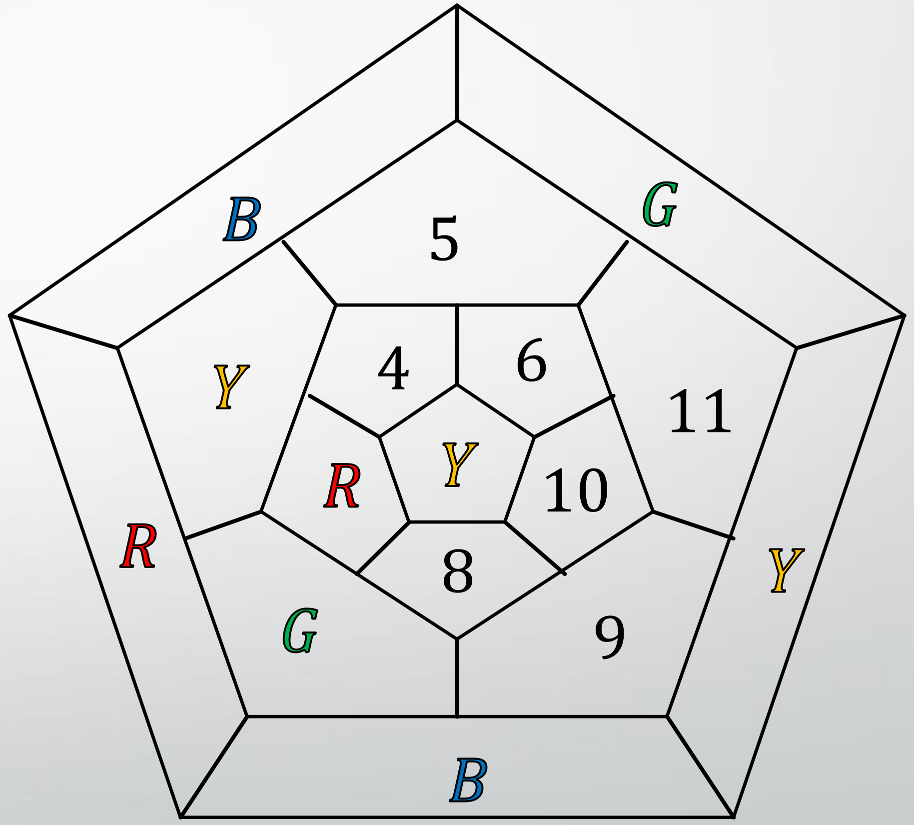

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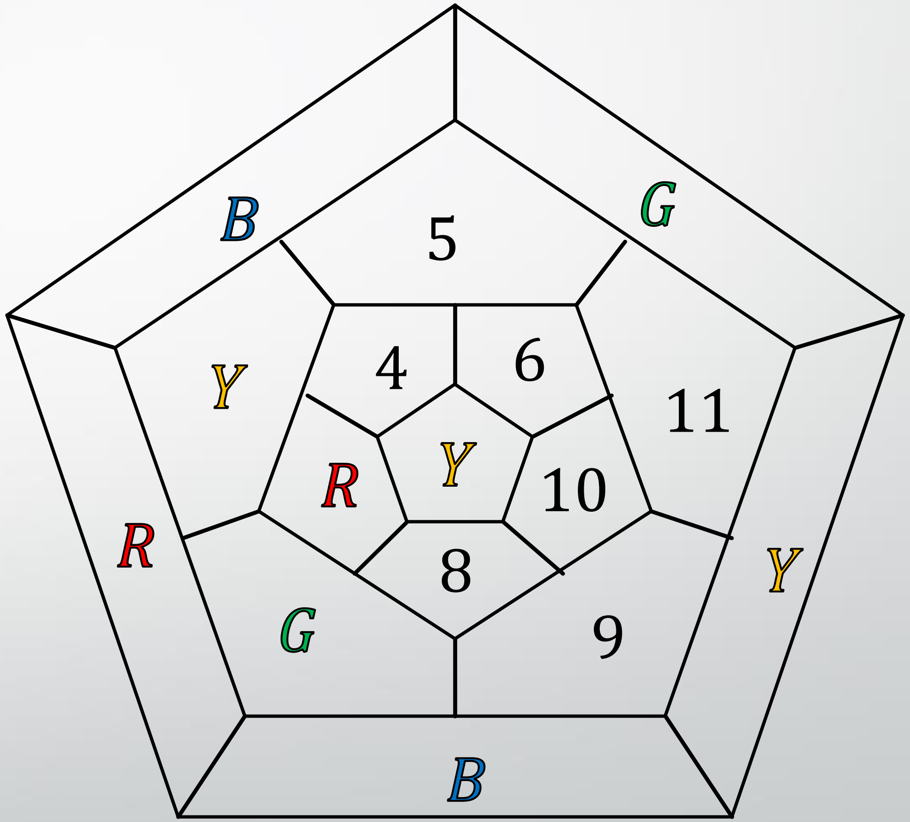

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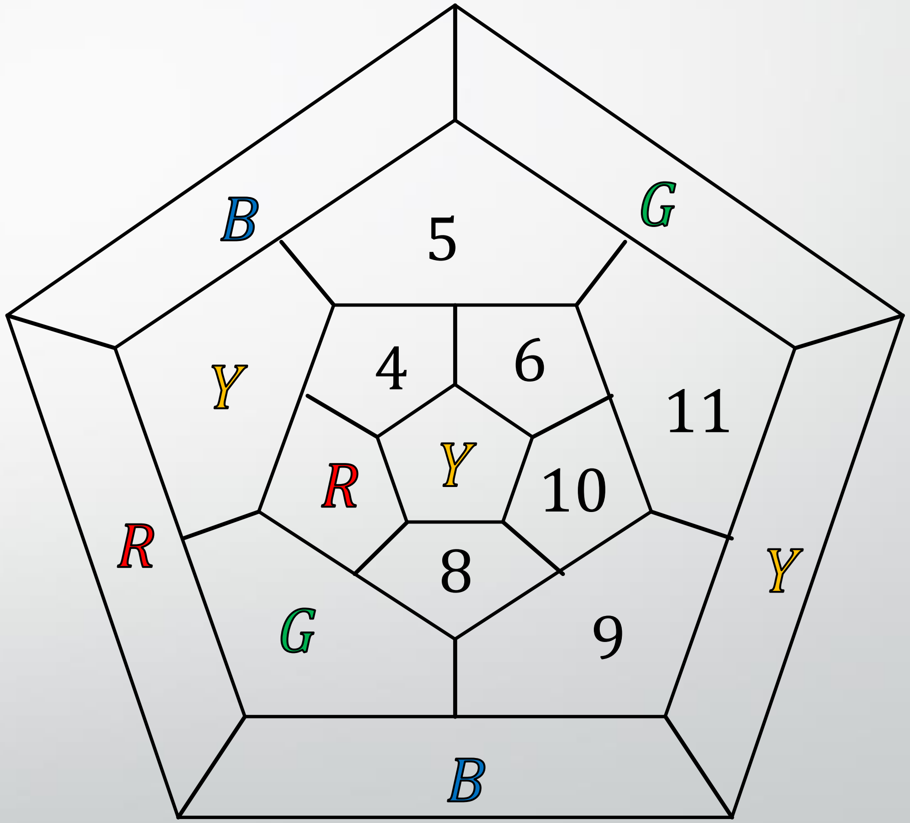

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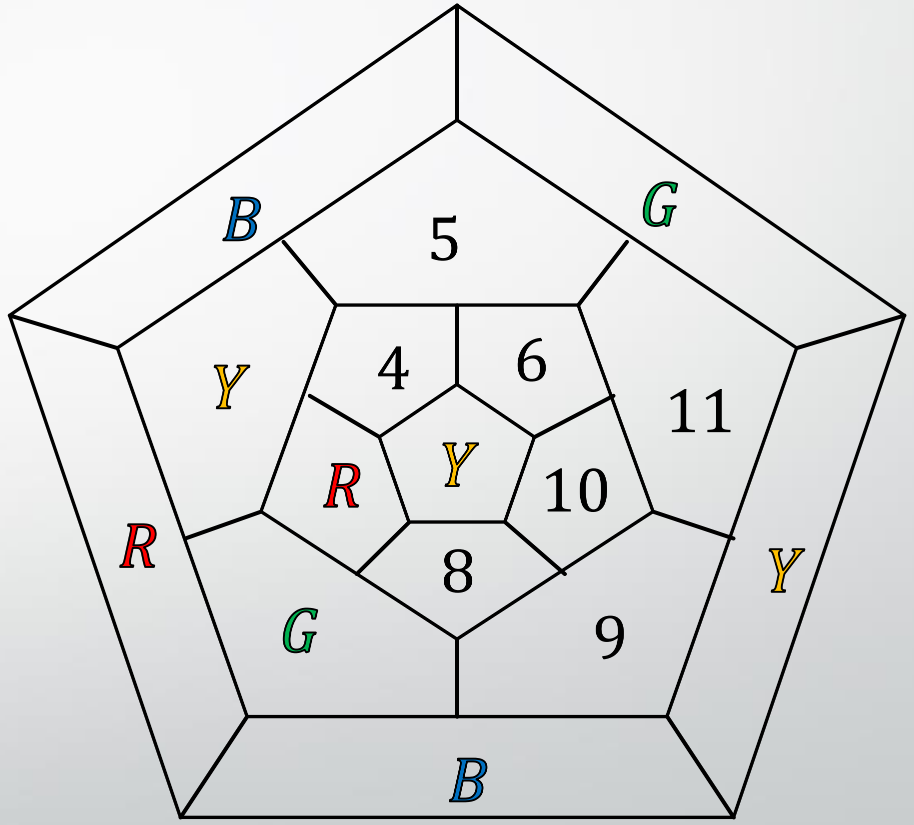
- Therefore, 7 cannot be Y. Thus, region 8 must be Y.



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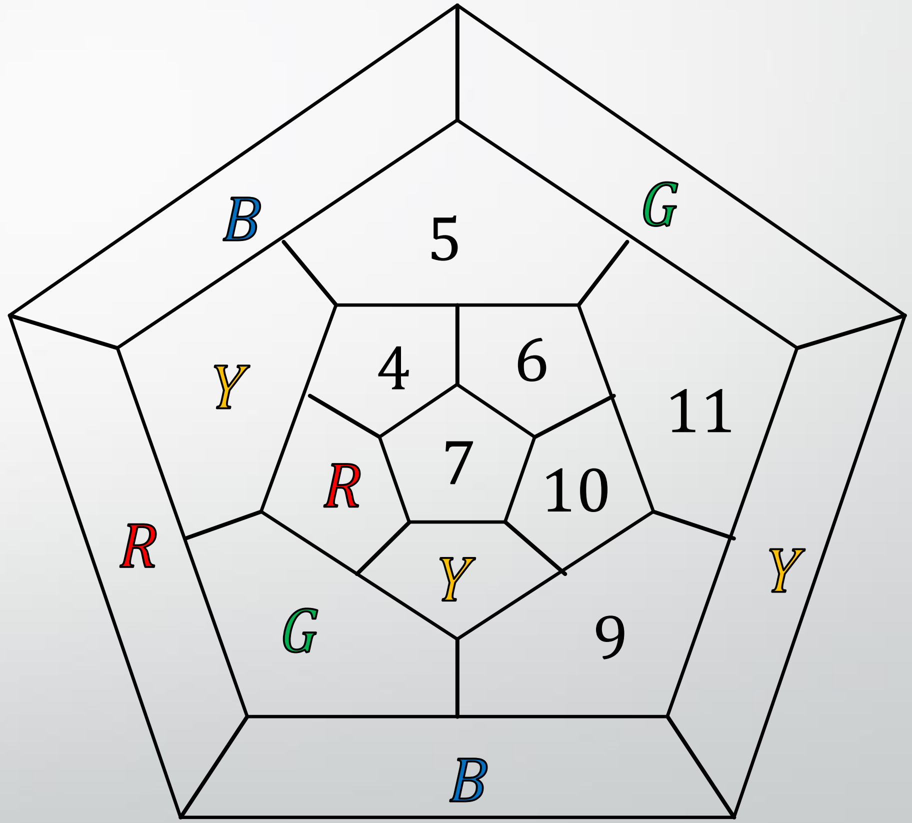

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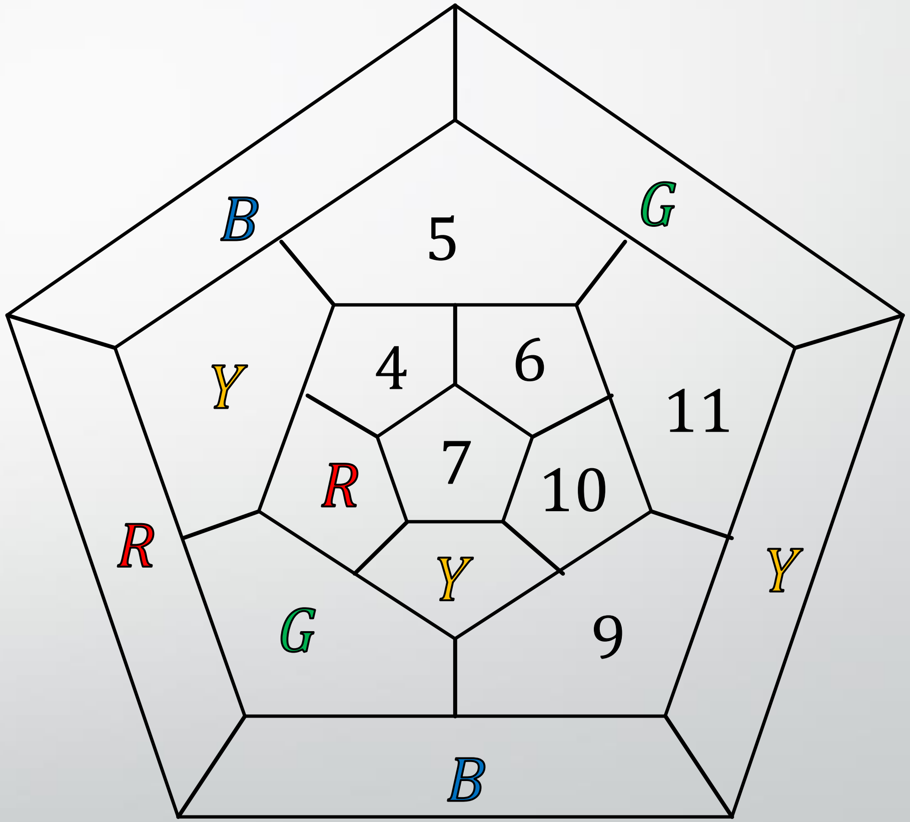

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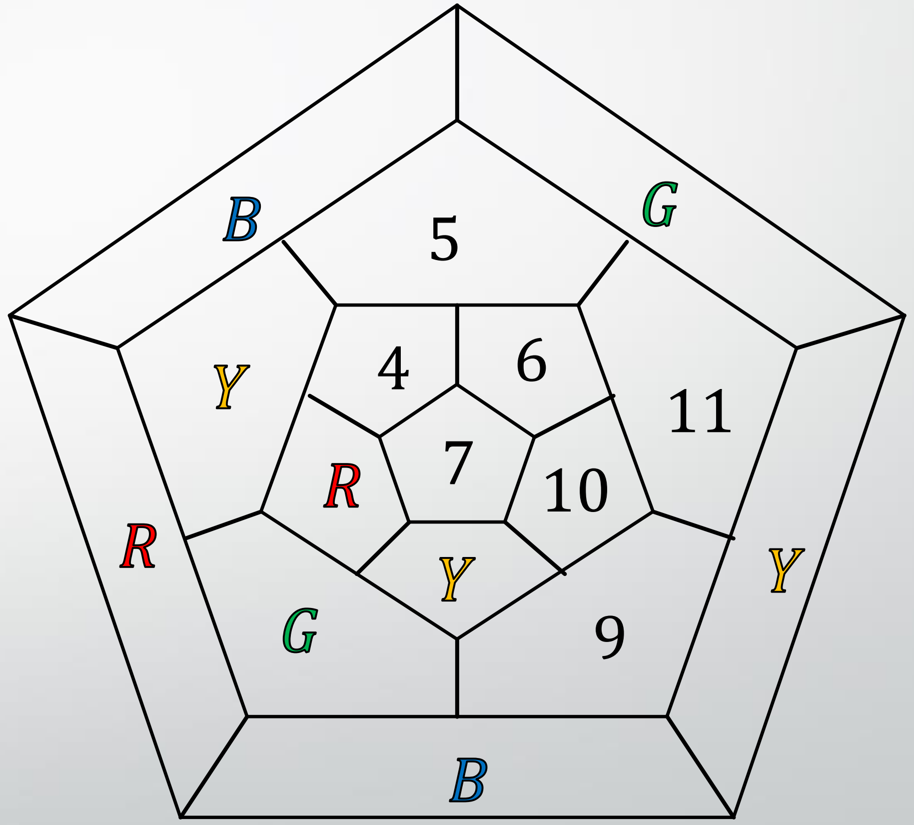
- The next region in the circuit must be 7, 10, or 9.
- 7 is already adjacent to a R region.



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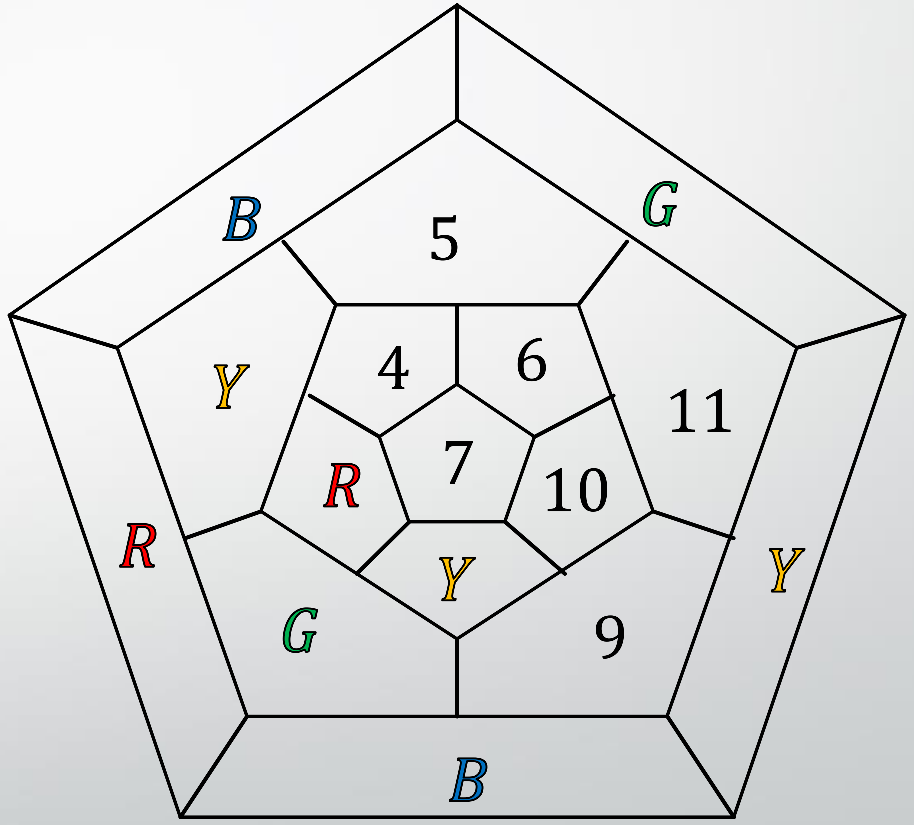

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- The next region in the circuit must be $7, 10$, or 9.
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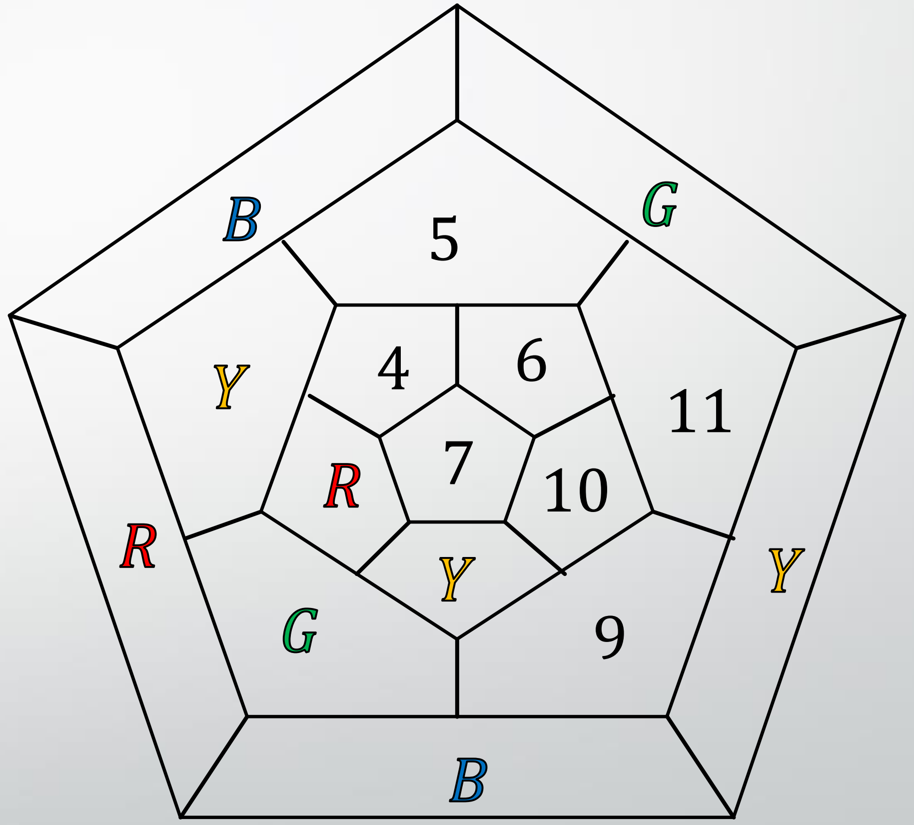
- Therefore, 9 must be R, and we complete the RY circuit.



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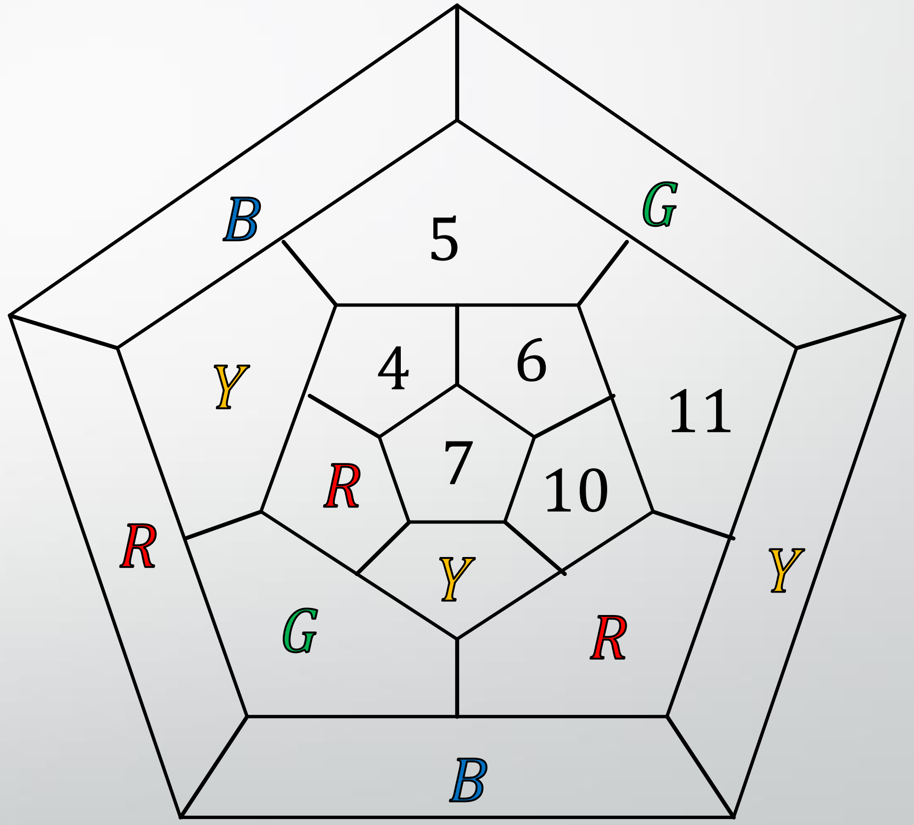

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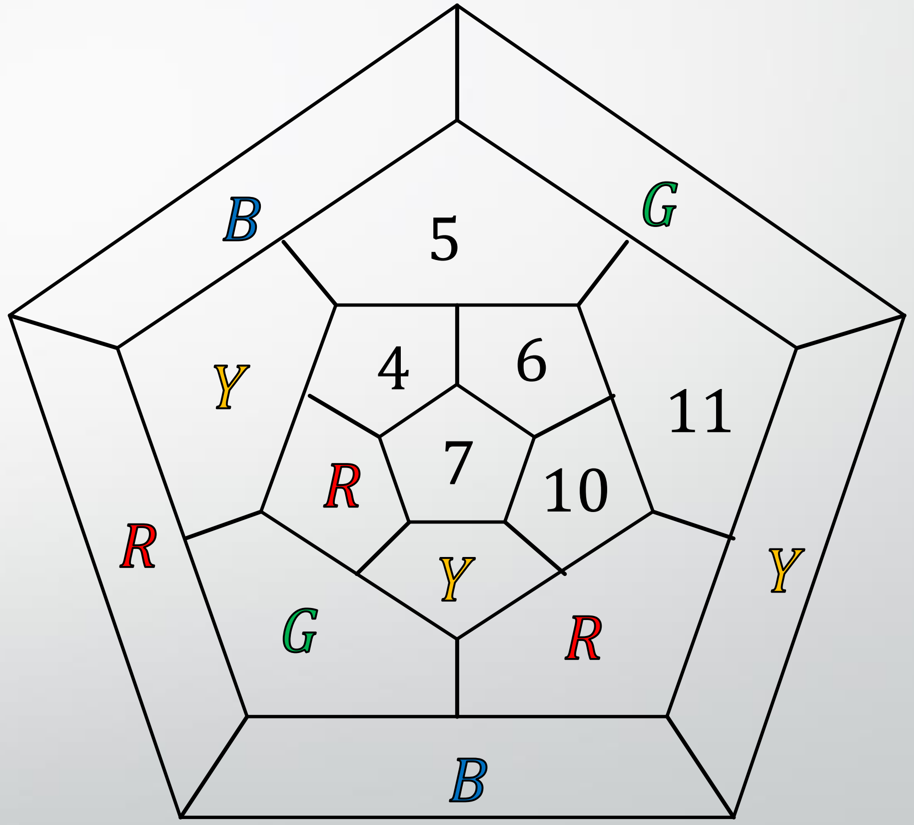

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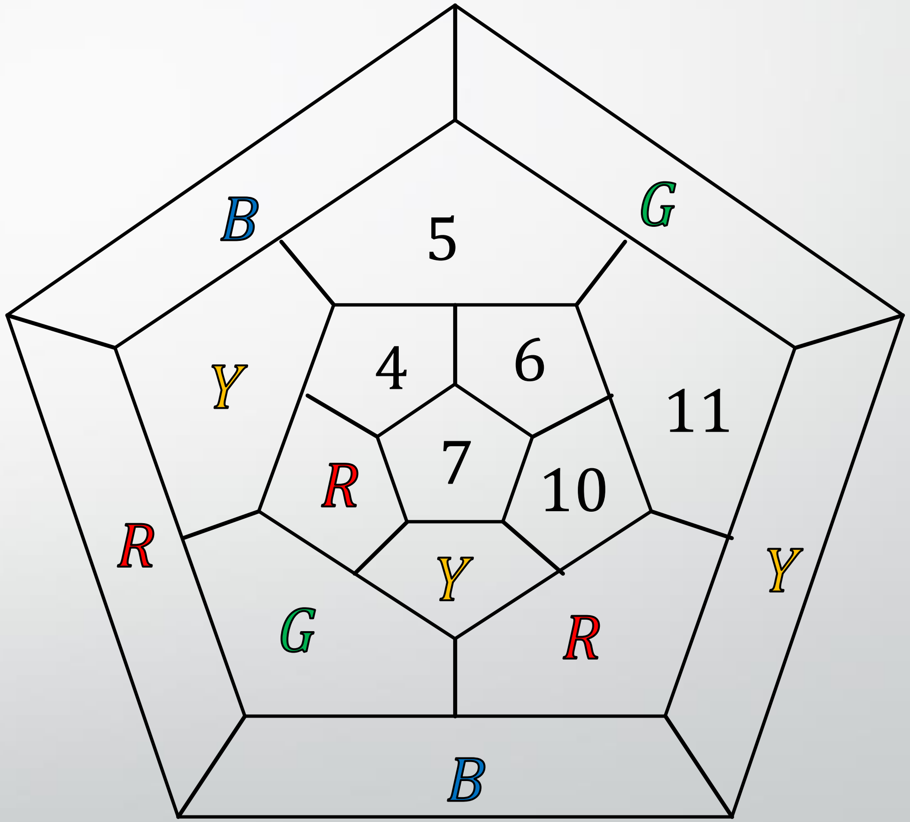
- Now we will form the RG circuit. It will help to color region 5 R now.



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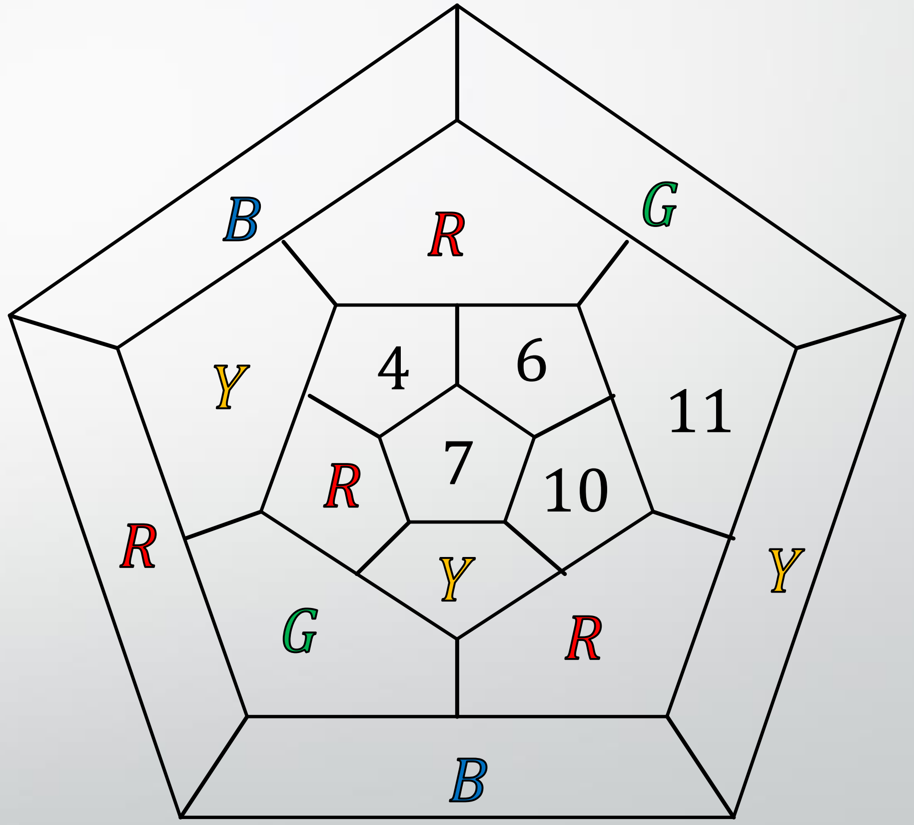

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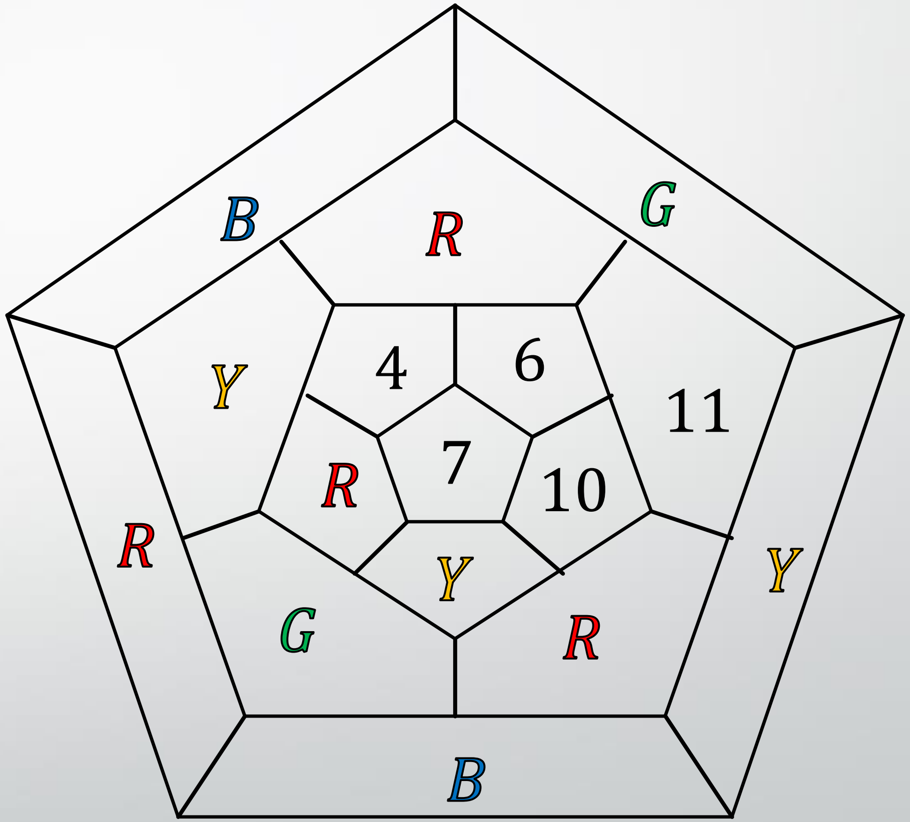

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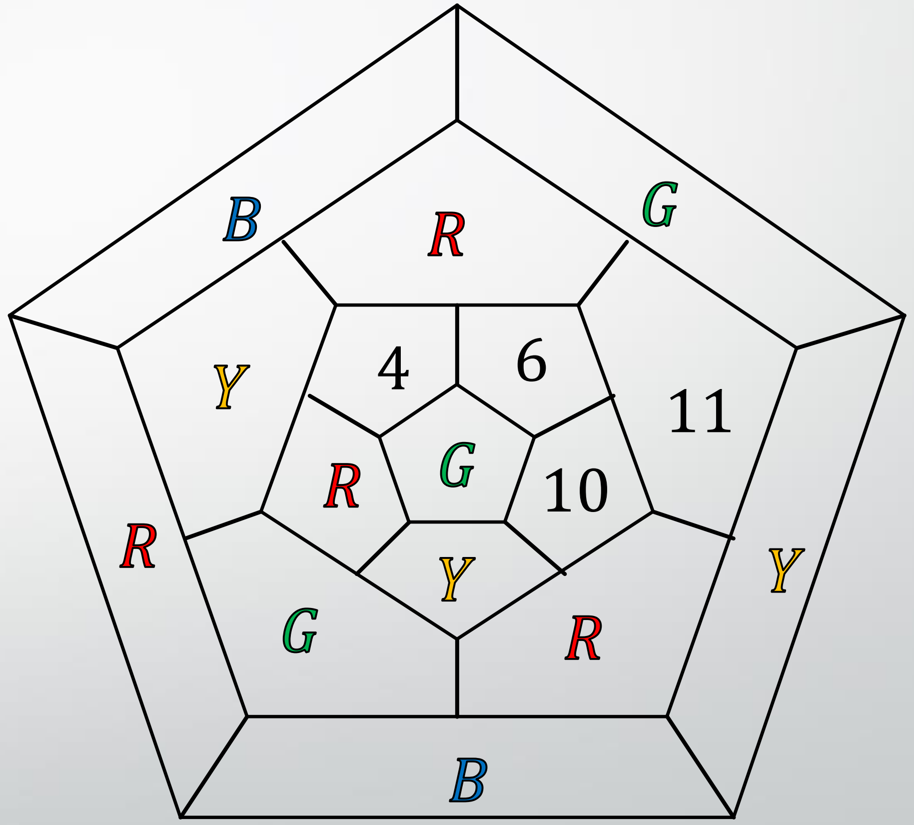

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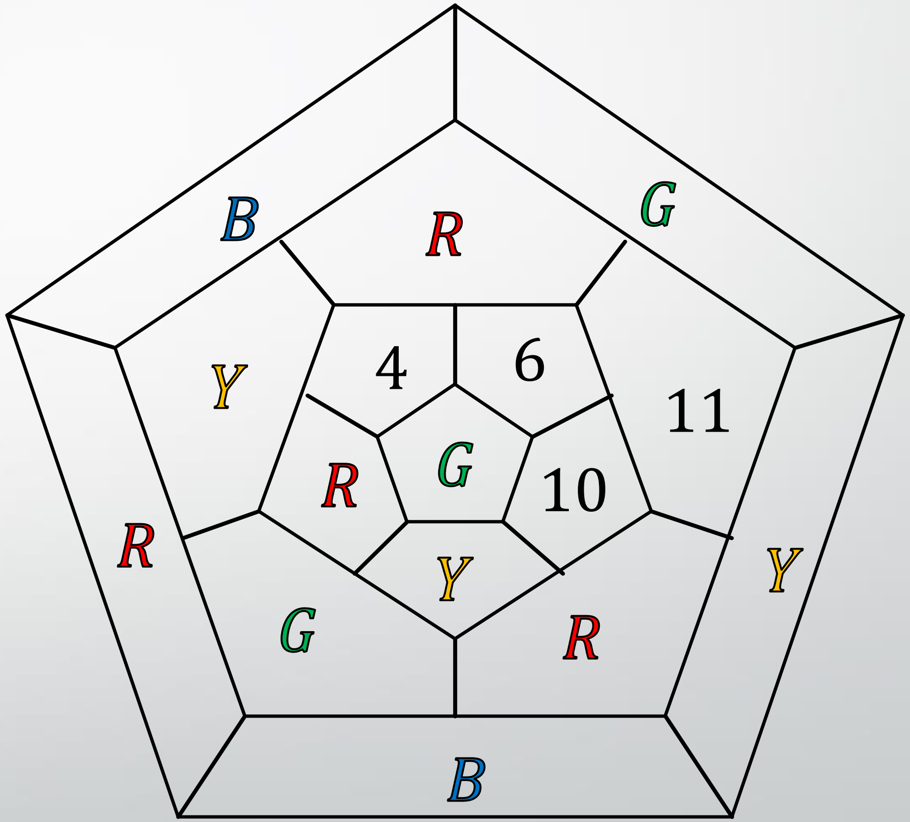

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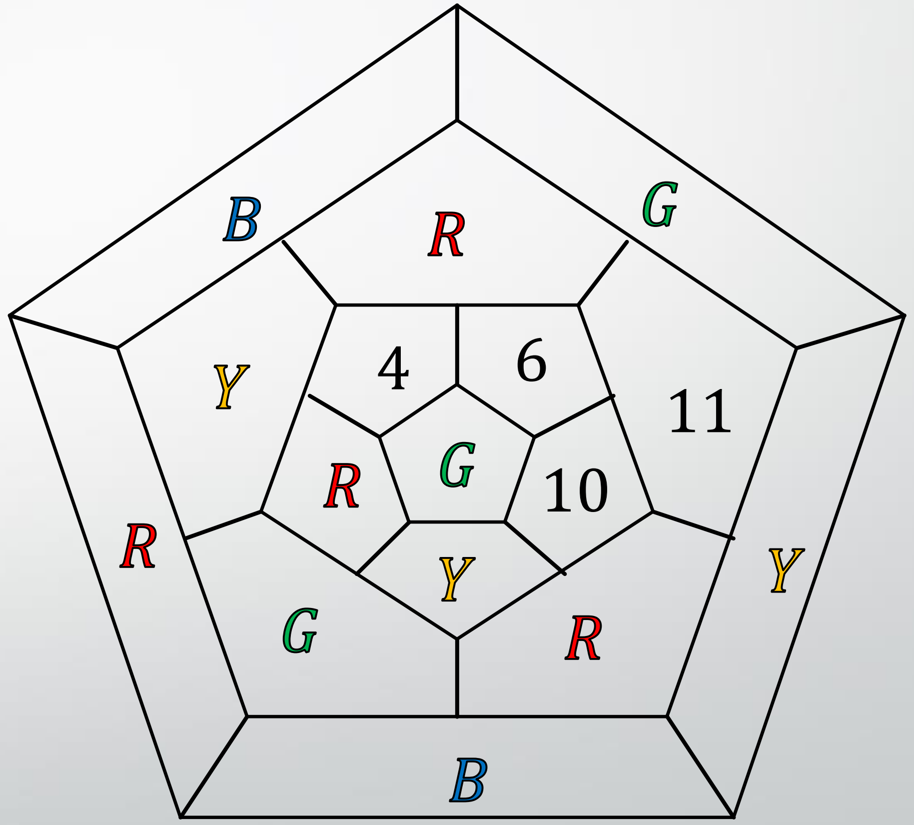
- Therefore, Region 4 is G.



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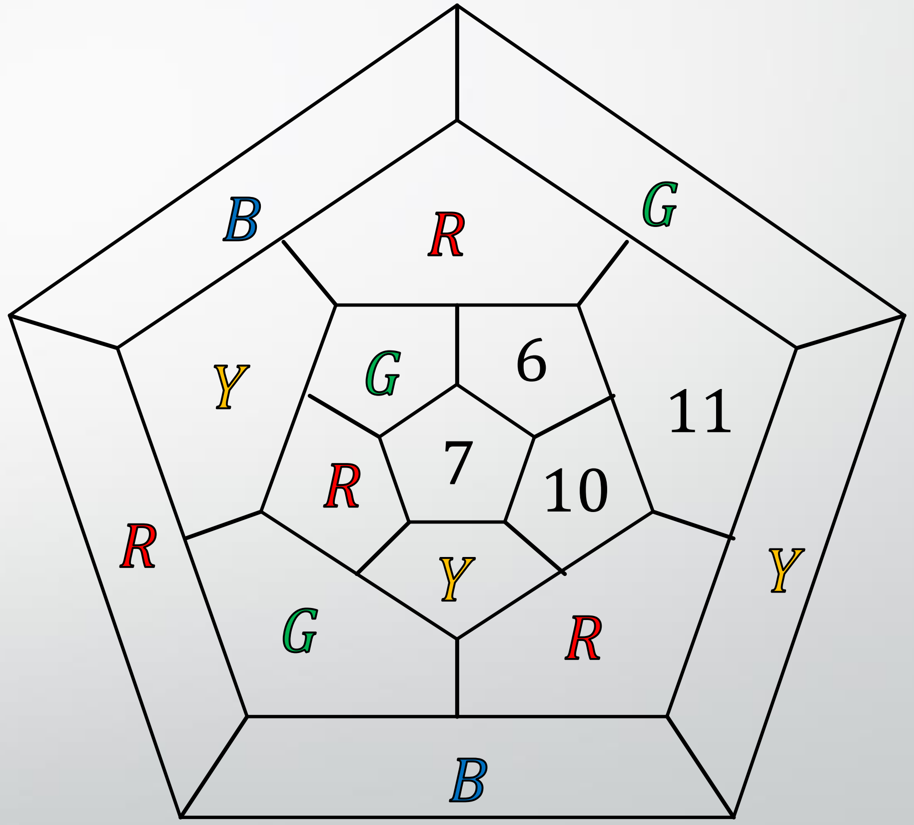

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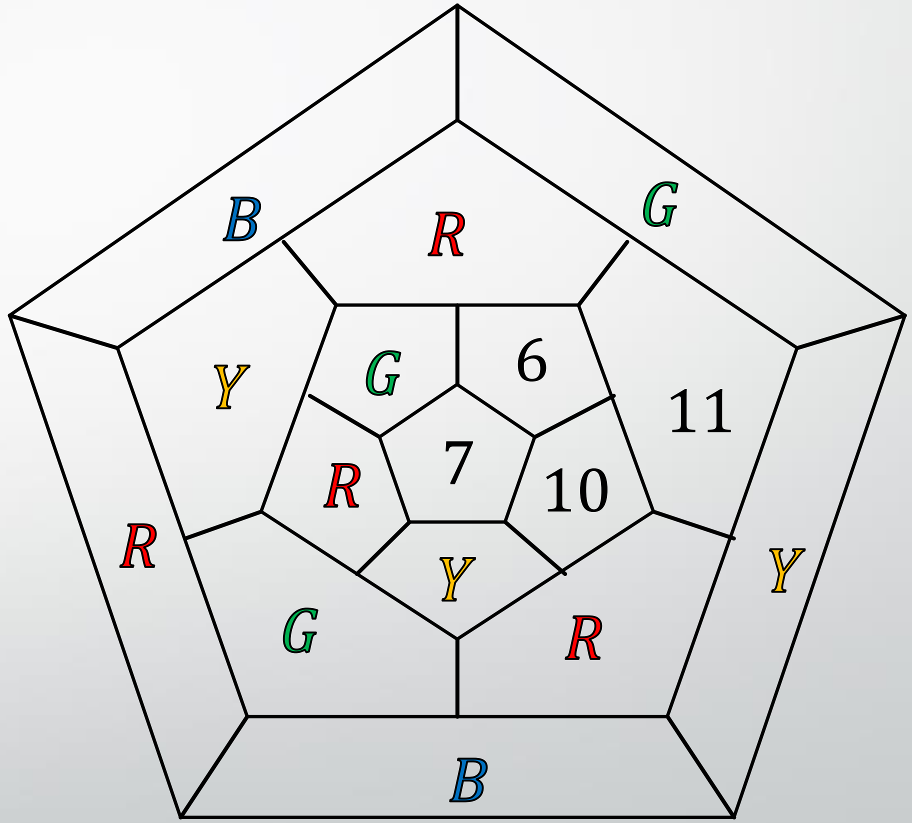
- Now our RG circuit is completed!



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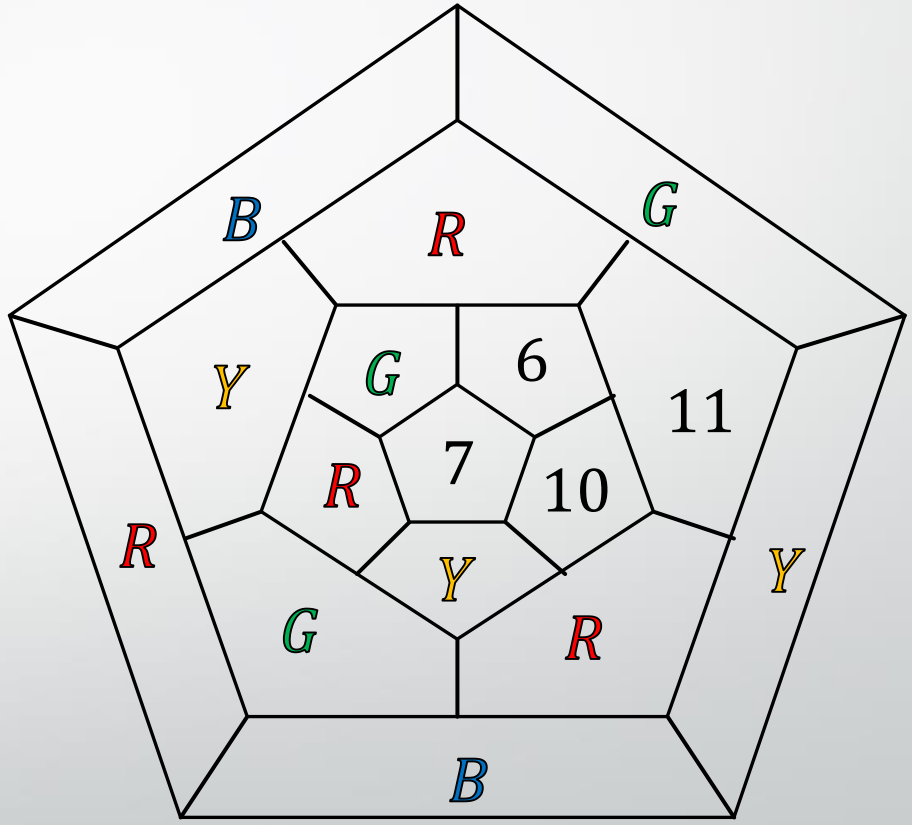

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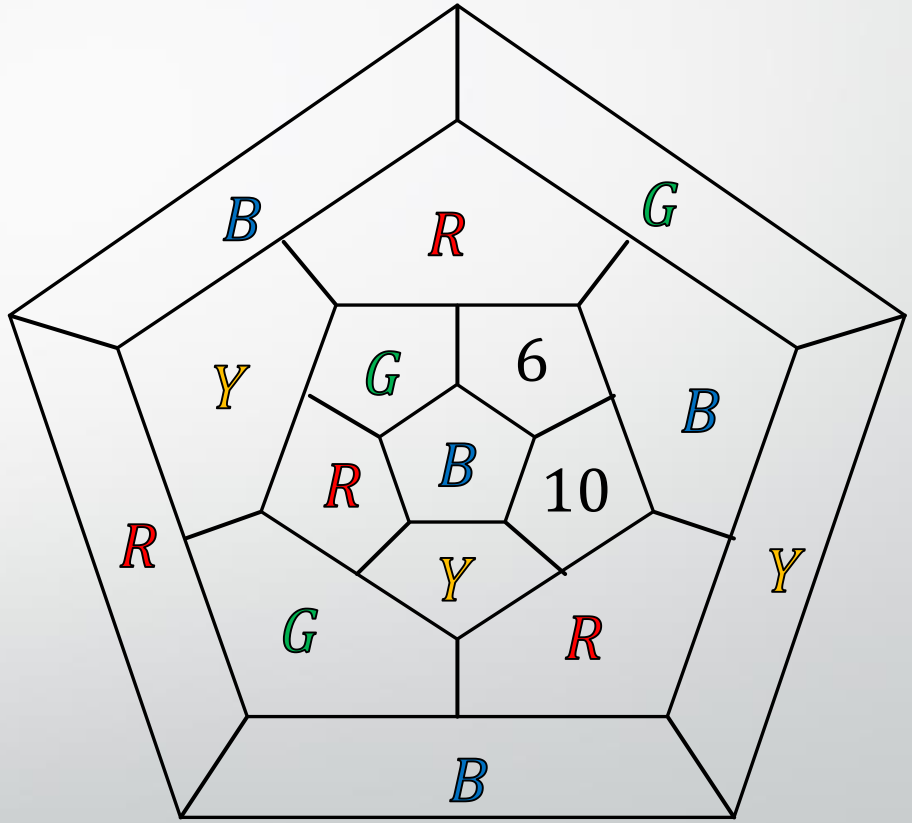

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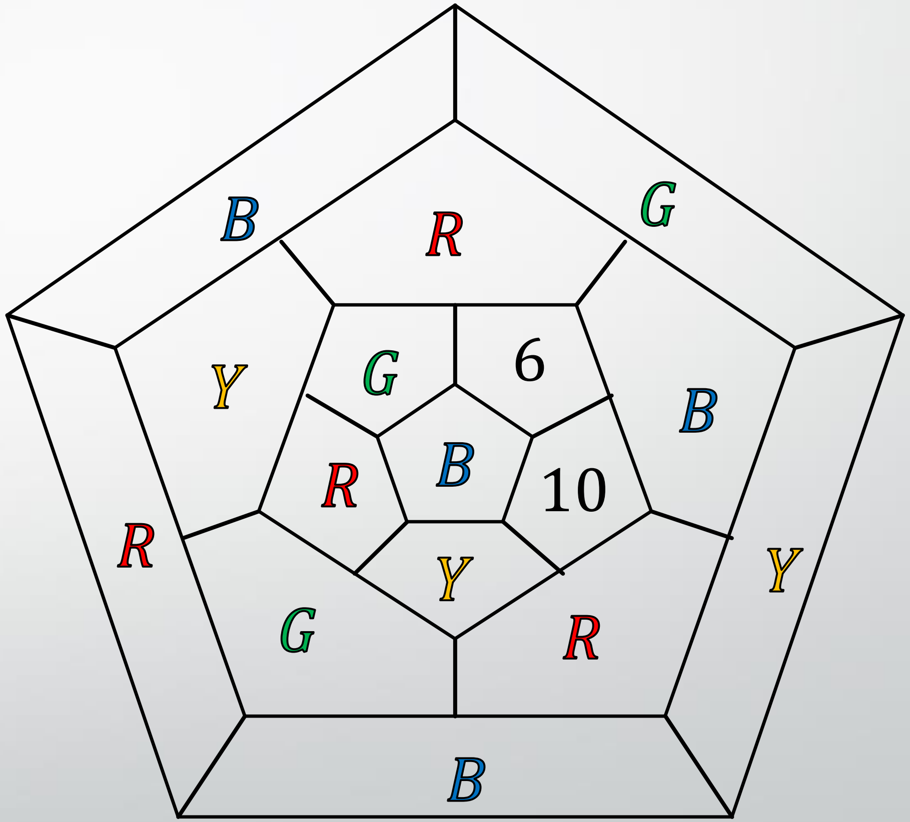

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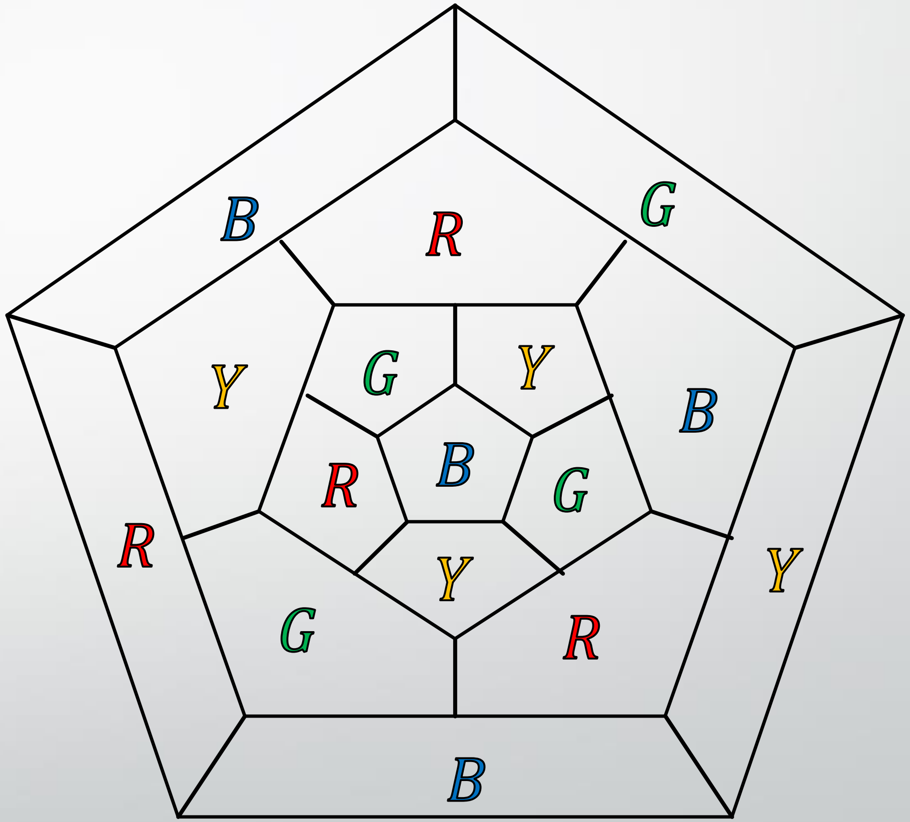

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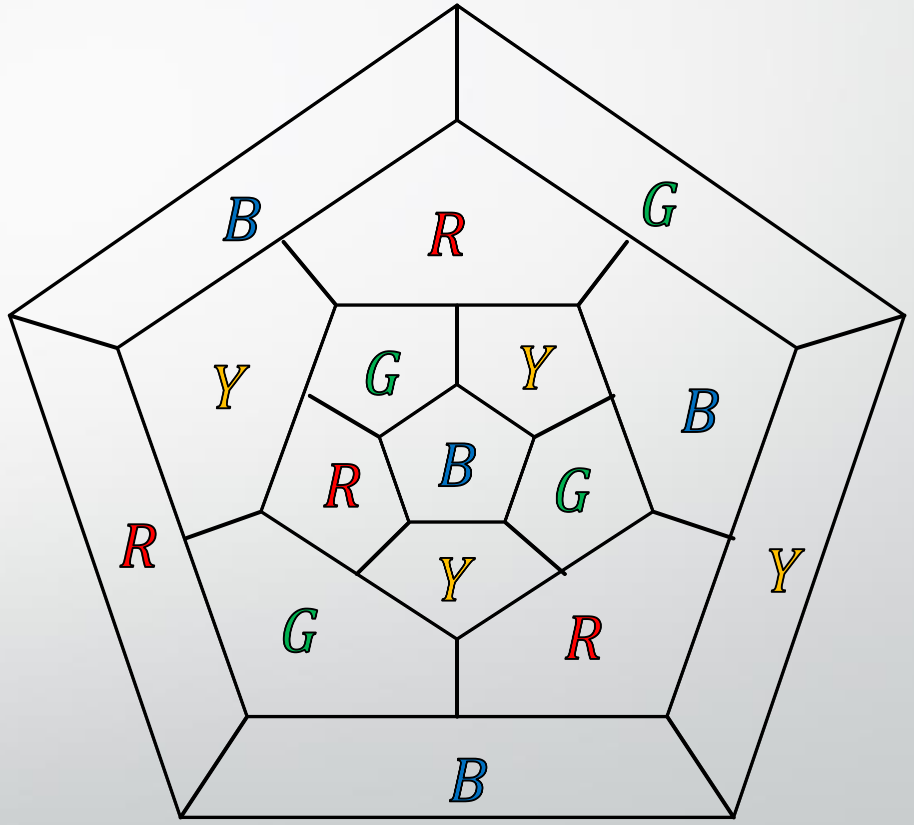
- We have determined colors for all interior regions of the map!



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- Otherwise, we completely explored all cases where the RY chain starts with the boundary region, then region 1, then region 3.
- After examining all cases, we obtain 4 colorings, corresponding to the original coloring c of the Errera map and the colorings $\alpha(c), \alpha^2(c)$, and $\alpha^3(c)$.

An Effective Algorithm

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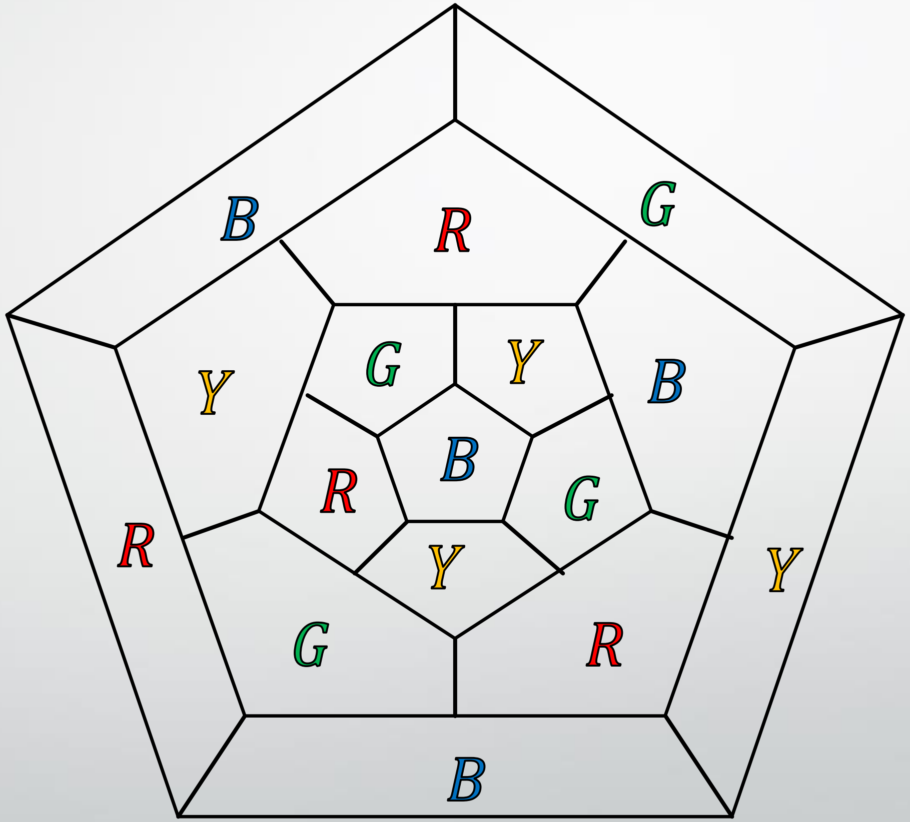
- **Result:** Impasse in an Errera Map with Holes can be resolved by resolving the impasse in the underlying Errera Map subgraph.
- Therefore, in each of the cases for which α fails to resolve impasse, ϵ or $\epsilon\alpha$ resolves impasse.

The Errera Map



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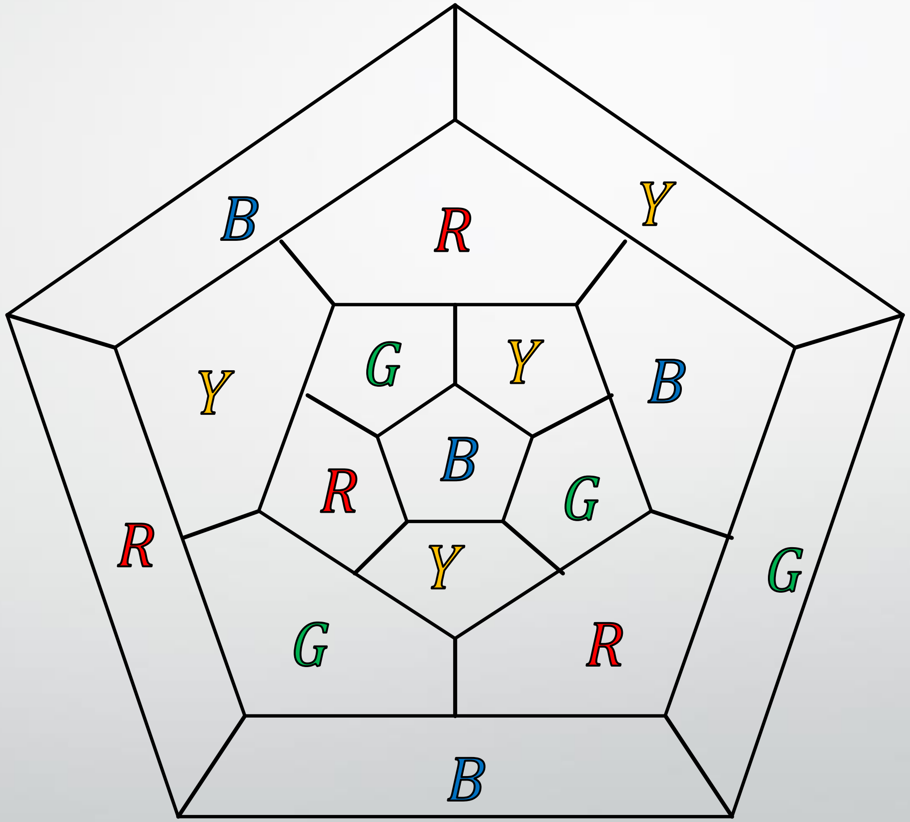

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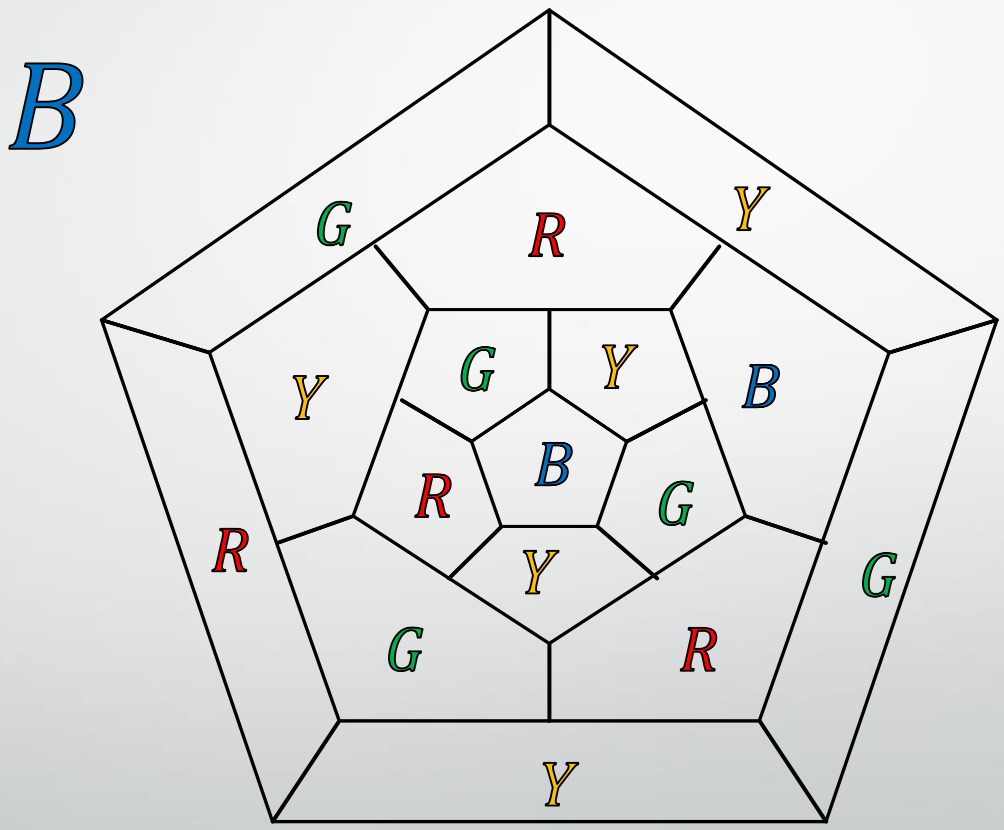

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  - If we exceed N iterations, or if we detect that we have returned to our original coloring,
    - Handle Errera Map cases by trying ϵ, then trying $\epsilon\alpha$.

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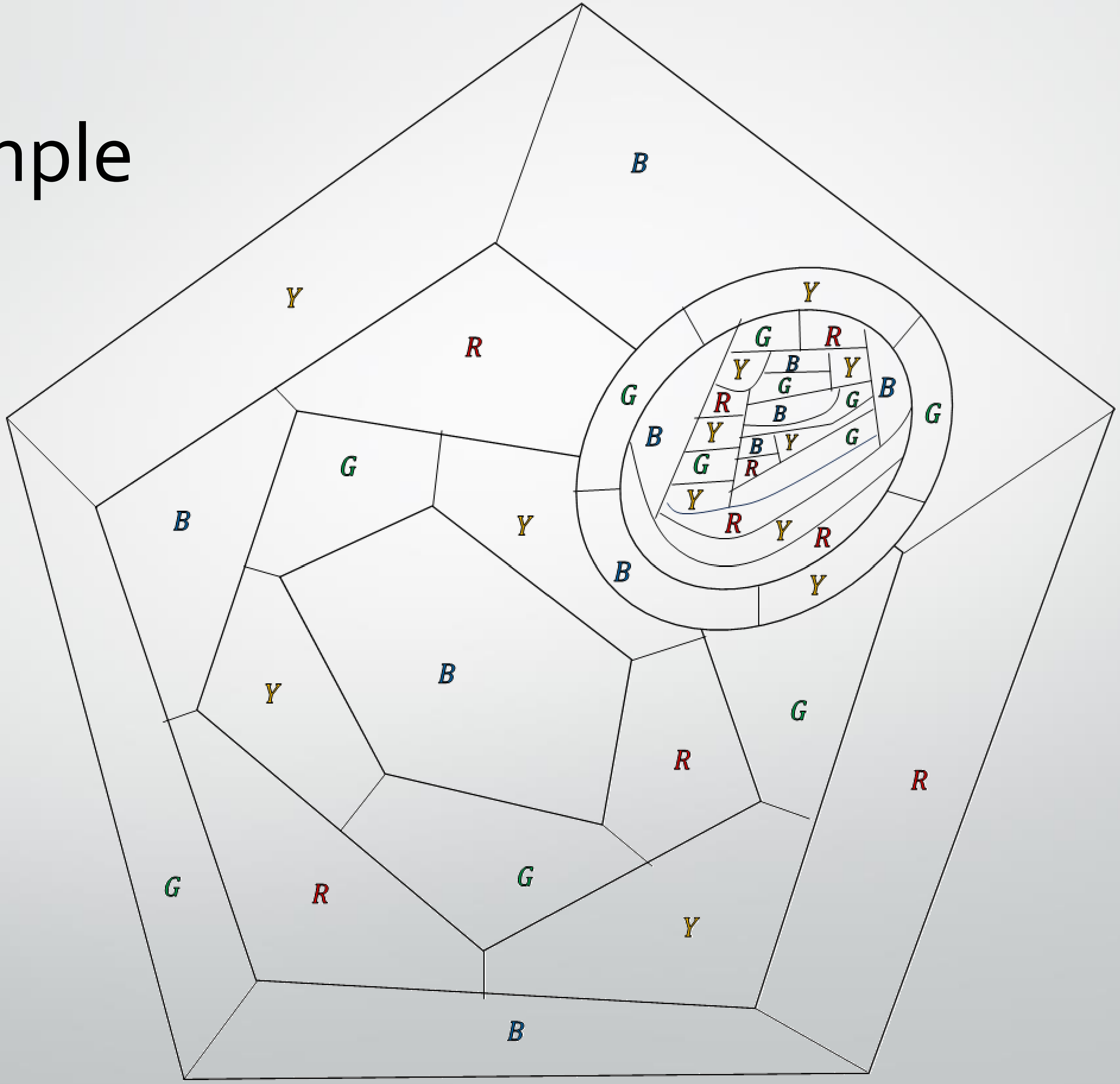
- In fact, up until last week, this had handled *all graphs.* But recently, we encountered the graph on the following slide.

A New Example



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  - This is an implementation problem as opposed to a conceptual problem, and can easily be addressed.
- The mirror image of this graph *would* be colored by the algorithm as is; therefore, it is possible that the mirror image has been encountered previously and gone unnoticed.

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- For all tested cases where repeated application of α doesn't resolve impasse, $\epsilon\alpha^n$ resolves impasse for some n.
  - This was initially motivated by the conjecture that all such graphs contain the Errera Map, although we now know this is not exactly the case.
- This leads to our final question:

For any impasse coloring c of a map M, is there some n such that either $\alpha^n(c)$ or $\epsilon\alpha^n(c)$ is not at impasse?

Similarly, can it be shown that the operations α and ϵ do not generate a group for any partial coloring?

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Thank you!