Geometric Analysis of Ricci Curvature

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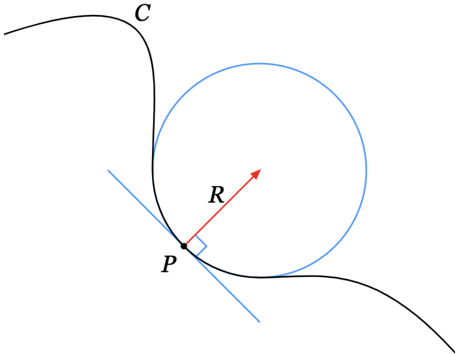
Outline

- (Very) Informal introduction to Curvature
- Curvature of Riemannian Manifolds
- Eigenvalue Problems
- Extension to Integral Curvature

Curvature

What is curvature?

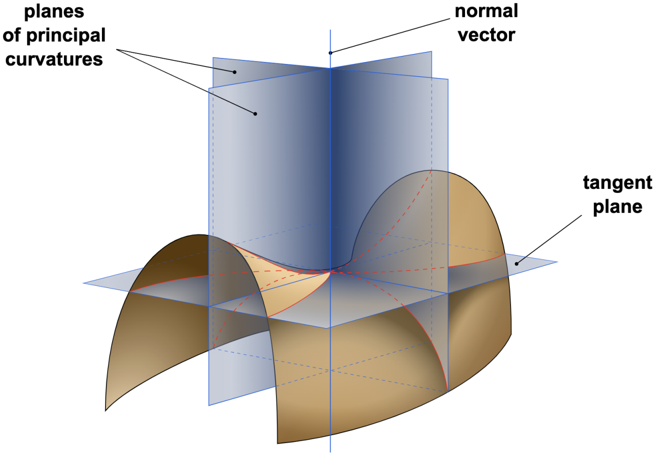
What is curvature? In one dimension,



$\kappa = \frac{1}{R}$.

Curvature

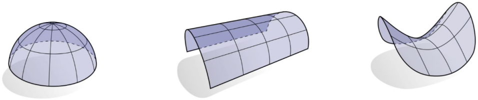
In two dimensions, we have **principal curvatures**



Curvature

Let K_1 and K_2 be the two principal curvatures.

- **mean curvature** $M = \frac{K_1+K_2}{2}$ (extrinsic)
- **Gauss curvature** $K = K_1 K_2$ (instrinsic)



$K > 0$ $K = 0$ $K < 0$

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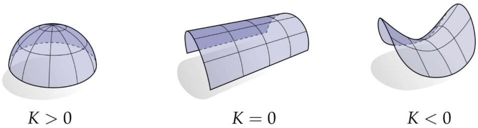

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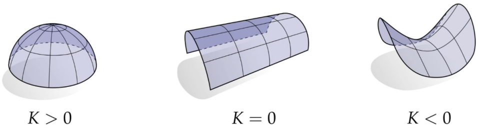

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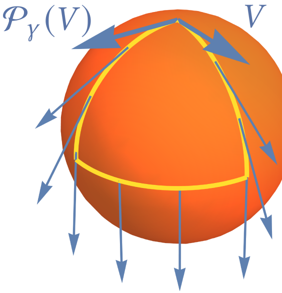
What about intrinsic?

Curvature

Characterizations of curvature: **non-commutativity**



Curvature

An Abrupt Introduction:
Let $\mathfrak{X}(M)$ be the space of vector fields on M.

Definition

Define the **Riemann Curvature Tensor**
$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$\begin{aligned} R(X, Y, Z) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z \end{aligned}$$

where $[X, Y] = XY - YX$ is the Lie bracket.

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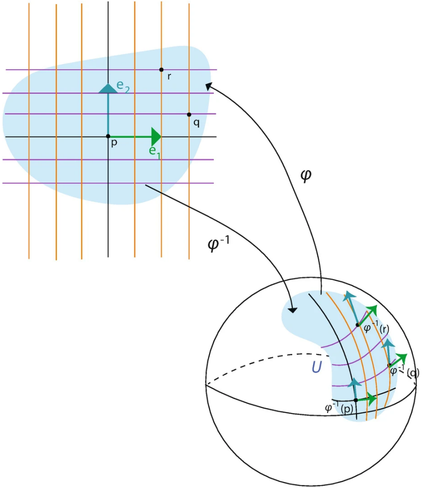
On $\mathbb{R}^n$,

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = (XY - YX) Z = \nabla_{[X,Y]} Z.$$

A **Riemannian manifold**

- Locally "looks like" $\mathbb{R}^n$, i.e., there are coordinates $(x^1, \ldots x^n)$.
- At each point there is an inner product space (tangent space) with inner product $g(e_i, e_j) = g_{ij}$ that depends smoothly on the point.
- Coordinate basis $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$.



Riemann Curvature Tensor

Using the metric, we have a rank $(4, 0)$ Riemann curvature tensor:

Definition

Let $X, Y, Z, W \in T_pM$ and $R(X, Y)Z \in T_pM$ be the $(3, 1)$-Riemann Curvature tensor. Then

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

In coordinates: $R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl}$

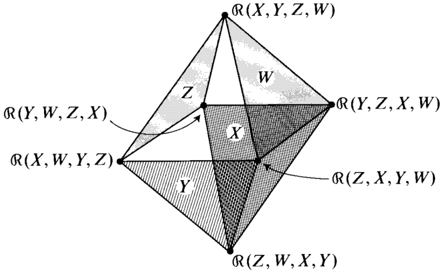
Riemann Curvature Tensor

Proposition (Symmetries of the Curvature Tensor)

1. $R(X, Y, Z, W) = -R(Y, X, Z, W)$
2. $R(X, Y, Z, W) = -R(X, Y, W, Z)$
3. $R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0.$

Corollary

If R is a quadrilinear mapping with the above properties, then $R(X, Y, Z, W) = R(Z, W, X, Y)$.



Parts of Riemann Curvature

Definition

For each 2-plane $P \subset T_pM$, the **sectional curvature** $K(P)$ is defined by

$$K(P) = R(X_1, X_2, X_2, X_1)$$

where $\{X_1, X_2\}$ is an orthonormal basis for P.

Remark

Using the symmetries of the curvature tensor, the sectional curvature fully determines the Riemann curvature tensor.

Remark

If M is a 2-dimensional Riemannian manifold, then the sectional curvature is equal to the Gauss curvature. (Recall that Gauss curvature is intrinsic).

Spaces of constant curvature

We say a Riemannian manifold has **constant curvature** κ if $K(P) = \kappa$ for all two-dimensional linear subspaces $P \subset T_pM$ and for all $p \in M$. If M is smooth, simply-connected and complete, then M is isometric to

- Euclidean space $\mathbb{R}^n$ when $\kappa = 0$.
- n-Sphere when $\kappa = 1$.
- Hyperbolic space when $\kappa = -1$.

Some results on positive sectional curvature

Theorem (Synge)

Let M be a compact even-dimensional orientable Riemannian manifold whose sectional curvatures are positive everywhere. Then M is simply connected.

Corollary

Let M be a compact even-dimensional Riemannian manifold of positive sectional curvature. Then either M is simply connected or $\pi_1(M) \cong \mathbb{Z}_2$.

Some results on negative sectional curvature

Theorem (Preissman)

If M is a compact Riemannian manifold with negative curvature, then any abelian subgroup of $\pi_1(M)$ different from the identity is infinite cyclic.

Example

$S^1 \times S^1$ does not carry a metric of negative curvature since $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$.

Ricci Curvature

Definition

The **Ricci curvature tensor** is the "trace of the Riemann tensor" or more precisely for $X, Y, Z \in T_pM$,

$$\mathrm{Ric}(Y, Z) := \mathrm{trace}(X \mapsto R(X, Y)Z).$$

If $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_pM, then

$$\mathrm{Ric}(Y, Z) := \sum_{i=1}^n R(e_i, Y, Z, e_i).$$

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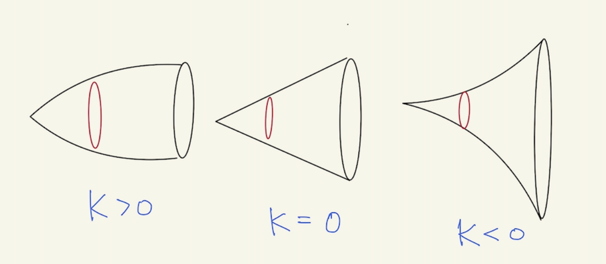
$$\mathrm{Ric}(Y, Z) := \sum_{i=1}^{n} R(e_i, Y, Z, e_i).$$

One geometric interpretation: **Bishop-Gromov volume comparison**: Let M_K^n be a complete n-dim. simply connected space of constant curvature K (hence $\mathrm{Ric} = (n-1)K$). Let $B_K(r)$ be the ball of radius r in M_K^n. Then

$$\mathrm{vol}(B(p, r)) \leq \mathrm{vol}(B_K(r)).$$

Ricci Curvature



Application of Ricci Curvature to Physics

Einstein's field equation

The equation that relates the geometry of spacetime to the distribution of matter within it:

$$\mathrm{Ric}_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = \kappa T_{ij},$$

- Ric_{ij} **Ricci tensor**
- $R = \mathrm{trace}(\mathrm{Ric})$ *scalar curvature*
- g_{ij} *metric tensor*
- Λ *cosmological constant*
- $\kappa = \frac{8\pi G}{c^4} \approx 2.076 \times 10^{-43} N^{-1}$ *Einstein's gravitational constant*
- T_{ij} *stress-energy tensor.*

Ricci curvature

Some more classical results with a lower bound on the Ricci curvature $(\mathrm{Ric}_{ij} \geq (n-1)Kg_{ij}, K \in \mathbb{R})$:

1. **Myers theorem**: If a complete Riemannian manifold has positive Ricci curvature, then its fundamental group is finite. The diameter is bounded by $\frac{\pi}{\sqrt{K}}$.
2. **Bochner vanishing**: If a compact n-dim. Riemannian manifold has $\mathrm{Ric} \geq 0$, then its first Betti number is at most n, equality if and only if $M = T^n$, n-torus.
3. **Cheeger-Gromoll Splitting**: If M is compact and $\mathrm{Ric} \geq 0$, then $\pi_1(M)$ contains a normal free abelian subgroup $A = \oplus_1^k \mathbb{Z}$, $k \leq \dim(M)$, of finite index in $\pi_1(M)$. Each $\mathbb{Z} \subset \pi_1(M)$ gives rise to an isometric splitting $\tilde{M} \cong \tilde{N} \times \mathbb{R}$ of the universal cover $\tilde{M}$.

Geometric Analysis with Ricci Curvature

Some geometric analysis tools:

- **Laplace-Beltrami operator**:

$$\Delta u = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j u)$$

- **Bochner formula**:

$$\frac{1}{2} \Delta |\nabla u|^2 = |\operatorname{Hess} u|^2 + g(\nabla u, \nabla(\Delta u)) + \operatorname{Ric}(\nabla u, \nabla u)$$

- **Mean Curvature Comparison**:

$$m \leq m_K$$

where m is the mean curvature of geodesic spheres and m_K is the mean curvature of geodesic spheres in M_K^n.

Eigenvalue problem

Consider the eigenvalue equation of the Laplacian on M:

$$\begin{cases} \Delta u = -\mu u \\ u \in H, \end{cases}$$

where H is some space of functions defined on M which will determine the "boundary condition".

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This is an extremal problem and can be characterized as

$$\mu_1 = \inf \left\{ \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} \mid u \in H, u \neq 0 \right\},$$

where the norm is the L^2 norm coming from the inner product $\langle f, g \rangle = \int_M fgdV$.
Note that a positive lower bound on μ_1 gives us the Poincaré inequalty:

$$\mu_1 \|u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2, \quad u \in H.$$

Estimates

On closed manifolds (compact, no boundary), by divergence theorem,

$$0 = \int_M div(\nabla u) dV = -\mu \int_M u dV$$

so we consider the space $H = \{u \in C^\infty(M) \mid \int_M u = 0, u \neq 0\}$.

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so we consider the space $H = \{u \in C^\infty(M) \mid \int_M u = 0, u \neq 0\}$. We have two optimal estimates

$$\begin{cases} \mu_1 \geq nK = \mu_1(S^n(1/\sqrt{K})), & \text{Ric} \geq (n-1)K > 0, & \text{(Lichnerowicz 1958)} \\ \mu_1(\Omega) \geq \frac{\pi^2}{D^2} = S^1(D/\pi), & \text{diam}(\Omega) = D, \text{Ric} \geq 0, & \text{(Zhong-Yang 1984)} \end{cases}$$

and the corresponding rigidity results:

$$\begin{cases} \mu_1 = nK, & M \text{ isometric to } S^n(1/\sqrt{K}) \text{ (Obata 1962)} \\ \mu_1 = \frac{\pi^2}{D^2}, & M \text{ isometric to } S^1(D/\pi) \text{ (Hang-Wang 2007)} \end{cases}$$

The two results have been generalized:

Theorem (Kröger (1998), Bakry-Qian (2000), Andrews-Clutterbuck (2013))

Let $\Omega \subset M^n$ be a bounded convex domain with diameter D and $\mathrm{Ric} \geq (n-1)K$. Then

$$\mu_1(\Omega) \geq \bar{\mu}_1(n, K, D),$$

where $\bar{\mu}_1(n, K, D)$ is the first non-trivial eigenvalue of

$$y'' - (n-1)\,\mathrm{tn}_K(x) y' + \lambda y = 0$$

on the interval $[-\frac{D}{2}, \frac{D}{2}]$ with Neumann boundary conditions.

$$\mathrm{tn}_K(x) = \begin{cases} \sqrt{K}\tan(\sqrt{K}x), & K > 0 \\ 0, & K = 0 \\ -\sqrt{-K}\tanh(\sqrt{-K}x), & K < 0. \end{cases}$$

Note if $K = 0$ then $\bar{\mu}_1 = \frac{\pi^2}{D^2}$ and if $D = \frac{\pi}{\sqrt{K}}$ then $\bar{\mu}_1 = nK$ (explicitly solved).

Sketch of Proof

The key relation between the Laplace operator and the Ricci curvature is given by the Bochner formula:

$$\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess} u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \operatorname{Ric}(\nabla u, \nabla u).$$

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Let u be the first eigenfunction on M and let y the first eigenfunction of the ODE. Let $f = y^{-1}(u(x))$ so that $\nabla f = \frac{\nabla u}{w'(f(x))}$. We can show $|\nabla f| \leq 1$. Choose a point $p, q \in M$ such that $w^{-1}(u(q)) = 0$ and $w^{-1}(u(p)) = D'$, where $D' \leq \mathrm{diam}(M)$. Let γ be a minimal geodesic connecting p and q. Then

$$\begin{aligned}\mathrm{diam}(M) &\geq \int_\gamma ds \geq \int_\gamma \frac{|\nabla u|}{|w'(w^{-1}(u))|} ds \\ &\geq \int_{w^{-1}(u(q))}^{w^{-1}(u(p))} dt = D'\end{aligned}$$

Integral Ricci Curvature

We now generalize the pointwise condition on curvature to an integral (average) condition.

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Let $\rho(x)$ be the smallest eigenvalue for the Ricci tensor and let $\rho_K := \max\{-\rho(x) + (n-1)K, 0\}$. Define the quantity

$$\bar{k}(p, K) := \left(\fint_M \rho_K^p \, dV \right)^{\frac{1}{p}}$$

which measures the amount of Ricci curvature lying below $(n-1)K$ in an L^p average sense.

Note that $\bar{k}(p, K) = 0$ iff $\mathrm{Ric} \geq (n-1)K$.

Integral Curvature

For $p > \frac{n}{2}$ and small $\bar{k}(p, K)$, many of the results extend (with error terms involving the L^p norm of Ricci curvature):

- Mean Curvature Comparison (not for maximal diameter)
- Volume Comparison
- Diameter Bound and finite fundamental group.
- Betti number Estimate

Also applications/interpretation in general relativity where small integral curvature corresponds to small quantum fluctuations in the energy-stress tensor.

Eigenvalue estimate under Integral Curvature

Some results on λ_1:

- For $K > 0$ and $p > \frac{n}{2}$, $\lambda_1 \geq nK(1 - C(n,p)\bar{k}(p,K))$(Aubry2007)

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- For $K > 0$ and $p > \frac{n}{2}$, $\lambda_1 \geq nK(1 - C(n,p)\bar{k}(p,K))$(Aubry2007)
- If $\operatorname{diam}(M) < D$, for any $\alpha \in (0,1)$ and $p > \frac{n}{2}$, there exists $\varepsilon(n,p,\alpha,D) > 0$ such that if $\bar{k}(p,0) < \varepsilon$, then $\lambda_1 \geq \alpha\frac{\pi^2}{D^2}$ (Ramos Olivé-S-Wei-Zhang 2019, S 2023)

Integral curvature

Recall that to prove Zhong-Yang estimate ($\lambda_1 \geq \frac{\pi^2}{D^2}$), we need to apply maximum principle.

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Recall that to prove Zhong-Yang estimate ($\lambda_1 \geq \frac{\pi^2}{D^2}$), we need to apply maximum principle.
Very roughly, we consider a function $Q = |\nabla u|^2 - z(u)$ and show that at its maximum point, $Q \leq 0$.

The first order condition is

$$0 = \nabla Q = \nabla |\nabla u|^2 - z'(u)\nabla u.$$

The second order condition gives us

$$0 \geq \Delta Q = \Delta |\nabla u|^2 - z''(u)|\nabla u|^2 - z'(u)\Delta u.$$

Applying Bochner formula, we immediately run in to an issue

$$0 \geq 2(|\operatorname{Hess} u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \operatorname{Ric}(\nabla u, \nabla u)) - z''(u)|\nabla u|^2 - z'(u)\Delta u.$$

The trick

The key idea in extending the point-wise bounds to integral curvature is to use an auxiliary function J and applying maximum principle to

$$Q_J = J|\nabla u|^2 - z(u).$$

By taking the Laplacian ΔQ_J, we get very roughly

$$(\Delta J - \tau \frac{|\nabla J|^2}{J} - 2J\rho_0)|\nabla u|^2 + J(\Delta Q) + C(u)\langle \nabla J, \nabla u \rangle$$

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By defining J to be the solution to $\Delta J - \tau \frac{|\nabla J|^2}{J} - 2J\rho_0 J = -\sigma J$, which can be transformed using $J = w^{-\frac{1}{\tau - 1}}$ to

$$\Delta w + Vw = \tilde{\sigma} w,$$

where $V = 2(\tau - 1)\rho_0$ and $\tilde{\sigma} = (\tau - 1)\sigma$, an eigenvalue equation for w. Then $J \to 1$ as $\rho_0 \to 0$.

Negative Curvature lower bound

Back to the pointwise Ricci lower bound situation, suppose $\mathrm{Ric} \geq -(n-1)K$, $K > 0$. We have an estimate by Yang

$$\mu_1 \geq \frac{\pi^2}{D^2} \exp\left(-\frac{1}{2} C_n \sqrt{(n-1)|K|} D\right)$$

where $C_n = \max\{\sqrt{n-1}, \sqrt{2}\}$.

Estimate requires delicate analysis of gradient estimate.

Future goal: Extend to integral Ricci curvature.

Thank you!