VERTEX ALGEBROIDS ASSOCIATED WITH CYCLIC LEIBNIZ ALGEBRAS WITH SMALL DIMENSIONS

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Abstract

Vertex algebroids play a vital role in the study of representation theory of vertex algebras. However, the classification of vertex algebroids is far from being complete. For this project, we classify vertex A-algebroids B associated with given cyclic non-Lie left Leibniz algebras B when B are 2- dimensional cyclic Leibniz algebras and 3-dimensional cyclic Leibniz algebras. For each given Leibniz algebra B, we use its algebraic structure and properties of vertex algebroid to construct a compatible unital commutative associative algebra A such that B is a vertex A-algebroid. As an application, we use the constructed vertex A-algebroids B to create a family of indecomposable non-simple vertex algebras V_B . In addition, we use the algebraic structure of the unital commutative associative algebras A that we found to study relations between certain types of vertex algebras V_B and the vertex operator algebra associated with the rank one Heisenberg algebra.

Background Materials

Definition 1: [DMS,FM] A left Leibniz algebra $\mathcal{L}$ is a C-vector space equipped with a bilinear map [,] : $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ satisfying the Leibniz identity $a_0(b_0c) = (a_0b)_0c + b_0(a_0c)$ for all a, b, c $\in \mathcal{L}$.

Definition 2: Let $\mathcal{L}$ be a left Leibniz algebra. $\mathcal{L}$ is cyclic if and only if there exists some $u \in \mathcal{L}$ such that $\mathcal{L} = \langle u \rangle = \mathrm{Span}\,\{u^k | k = 1,2, \dots\}$. If $\mathcal{L} = \langle v \rangle$, we call v a generator of $\mathcal{L}$.

Definition 3: [GMS] A 1-truncated conformal algebra is a graded vector space $C = C_0 \oplus C_1$ equipped with a linear map $\partial: C_0 \to C_1$ and bilinear operations $(u, v) \to u_i v$ for $i = 0{,}1$ of degree $-i - 1$ on $C = C_0 \oplus C_1$ such that the following axioms hold:

Derivation for $a \in C_0$, $u \in C_1$,

$$(\partial a)_0 = 0,\ (\partial a)_1 = -a_0,\ \partial(u_0 a) = u_0 \partial(a);$$

Commutativity for $a \in C_0$, $u, v \in C_1$,

$$u_0 a = -a_0 u,\ u_0 v = -v_0 u + \partial(u_1 v),\ u_1 v = v_1 u;$$

Associativity for $\alpha, \beta, \gamma \in C_0 \oplus C_1$,

$$\alpha_0 \beta_i \gamma = \beta_i \alpha_0 \gamma + (\alpha_0 \beta)_i \gamma$$

Background Materials

Definition 4: [LiY] Let $(A, *)$ be a unital commutative associative algebra and let B be a module for A as a nonassociative algebra. Then a vertex A-algebroid structure on B exactly amounts to a 1-truncated conformal algebra structure on $C = A \oplus B$ with $a, a' \in A$, $u, v \in B$ such that

- $a \cdot (a' \cdot u) - (a * a') \cdot u = (u_0 a) \cdot \partial(a') + (u_0 a') \cdot \partial(a)$
- $u_0(a \cdot v) - a \cdot (u_0 v) = (u_0 a) \cdot v$
- $u_0(a * a') = a * (u_0 a') + (u_0 a) * a'$
- $a_0(a' \cdot v) = a' * (a_0 v)$
- $(a \cdot u)_1 v = a * (u_1 v) - u_0 v_0 a$
- $\partial(a * a') = a \cdot \partial(a') + a' \cdot \partial(a)$

Definition 5: Let I be a subspace of a vertex A-algebroid B. The vector space I is called an *ideal* of the vertex A-algebroid B if I is a left ideal of the left Leibniz algebra B and $a \cdot u \in I$ for all $a \in A$, $u \in I$.

We set $A\,\partial(A) \coloneqq Span\{a \cdot \partial(a') | a, a' \in A\}$. The vector space $A\partial(A)$ is an ideal of the vertex A-algebroid B. Moreover, $A\partial(A)$ is an abelian Lie algebra. Observe that for $a, a', a'' \in A$, $u \in B$, we have

$\left(a \cdot \partial(a')\right)_0 a'' = 0$, and $(a \cdot \partial(a'))_0 u = a \cdot (u_0 \partial(a')) + (u_0 a) \cdot \partial(a') + \partial(u_1(a \cdot \partial(a')) \in A\partial(A)$

Proposition 1: Let A be a unital commutative associative algebra with the identity 1_A. Let B be a vertex A-algebroid such that B is a cyclic non-Lie left Leibniz algebra, $B \neq A\partial(A)$, and dim(B)=n. Then, there exists $b \in B$ such that $\{b, b_0 b, \dots, (b_0)^{n-1} b\}$ is a basis for B. We set $a = b_1 b$. Assume that $\mathrm{Ker}(\partial) = C1_A$. Then A is a local algebra with a basis $\{1_A, a, b_0 a, \dots, (b_0)^{n-2} a\}$.

Sketch proof of proposition 1: We show $\{1_A, a, b_0 a, \dots, (b_0)^{n-2} a\}$ is linearly independent. We set $\alpha 1_A + \alpha_0 a + \alpha_1 b_0 a + \cdots + \alpha_{n-2} (b_0)^{n-2} a = 0$. We then apply ∂ to this equation and the result is $\alpha_0 \partial(a) + \alpha_1 \partial(b_0 a) + \cdots + \alpha_{n-2} \partial\left((b_0)^{n-2} a\right) = 0$.

In an earlier discovery, we know that $\{\partial(a), \partial(b_0 a), \dots, \partial\left((b_0)^{n-2} a\right)\}$ is linearly independent since it is a basis for $\partial(A)$. This allows us to conclude $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-2} = 0$. Also, since we have $\alpha 1_A = 0$ we know $\alpha = 0$ and $\{1_A, a, b_0 a, \dots, (b_0)^{n-2} a\}$ is linearly independent. In this section, we also find $A =$

Methodology

Classification of vertex algebroids associated with non-Lie left Leibniz algebras

2-dimensional case

For a non-Lie Leibniz algebra B such that dim B=2, B is isomorphic to a cyclic left Leibniz algebra generated by b with either $b_0(b_0 b) = 0$ or $b_0(b_0 b) = b_0 b$. For theorem 1 below, we focus on $b_0(b_0 b) = 0$.

Theorem 1: Let A and B be defined as in proposition 1 with dim B=2, $B \neq A\partial(A)$, and $\mathrm{Ker}(\partial) = C1_A$. Clearly, there exists $b \in B$ such that $\{b, b_0 b\}$ is a basis for B. We set $a = b_1 b$, the set $\{b, \partial(a)\}$ is a basis of B and the set $\{1_A, a\}$ is a basis for A. We assume that $b_0(b_0 b) = 0$. We found the following: $b_0 a = 0$, $a \cdot b \in \partial(A)$, $a * a = 0$ with $A \cong C[x]/(x^2)$, and $a \cdot \partial(a) = 0$.

In addition, B is a module of A as a commutative associative algebra. Therefore, the ideal (a) is the unique maximal ideal of A. Moreover, for $u \in (a)$, $w \in B/A\partial(A)$, $w_0 u = 0$ and $u \cdot w = 0$.

Sketch proof of theorem 1:

Assume that $b_0(b_0 b) = 0$. Since $\partial(a) = 2 b_0 b$, we have $b_0 \partial(a) = 0$. Since $\partial(b_0 a) = b_0 \partial(a) = 0$, we have that $b_0 a \in \mathrm{Ker}(\partial) = C1_A$. Thus, there exists $\lambda \in C$ such that $b_0 a = \lambda 1_A$. Moreover, $b_0 b_0 a = 0$.

We set $a \cdot b = \beta_1 b + \beta_2 \partial(a)$, $a \cdot \partial(a) = \frac{1}{2} \alpha_2 \partial(a)$.

Since $b_0(a \cdot b) = a \cdot b_0 b + (b_0 a) \cdot b = \frac{1}{2} a \cdot \partial(a) + \lambda b = \frac{1}{4} \alpha_2 \partial(a) + \lambda b$. Also, we have

$b_0(a \cdot b) = b_0\left(\beta_1 b + \beta_2 \partial(a)\right) = \beta_1 \frac{1}{2} \partial(a) + \beta_2 \partial(b_0 a) = \beta_1 \frac{1}{2} \partial(a)$.

This shows us that $\beta_1 \frac{1}{2} \partial(a) = \frac{1}{4} \alpha_2 \partial(a) + \lambda b$. Since $\{b, \partial(a)\}$ is linearly independent, we can conclude that $\lambda = 0$ and $b_0 a = 0$ as well as other relations.

3-dimensional case

There are 4 types of cyclic non-Lie Leibniz algebras of the 3-dimensional case. Here we show only the case when $x_0 x = y$ and $x_0 y = z$.

Theorem 2: Let A and B be defined as above with dim B=3, $B \neq A\partial(A)$, and $\mathrm{Ker}(\partial) = C1_A$. Then there exists $b \in B$ such that $\{b, b_0 b, (b_0)^2 b\}$ is a basis of B such that $(b_0)^3 b = 0$. Equivalently, $\{b, \partial(a), \partial(b_0 a)\}$ is a basis of B such that $b_0 \partial(b_0 a) = 0$. In addition, if we set $a = b_1 b$, then $\{1_A, a, b_0 a\}$ is a basis of A.

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Theorem 2 continued:

We assume $b_0 \partial(b_0 a) = 0$.

Then $\beta = 0$, $a * (b_0 a) = 0$, $a * a = (\gamma_1 + 1)\chi 1_A + \gamma_0 b_0 a$, $a \cdot b = \gamma_0 \partial(a) + \gamma_1 \partial(b_0 a)$, $a \cdot \partial(a) = \frac{1}{2} \gamma_0 \partial(b_0 a)$, and $\gamma_0 \chi = 0$.

If $\gamma_0 = 0$, we have $(b_0 a) \cdot b = 0$, $\chi = 0$, $a * a = 0$, $a \cdot \partial(a) = 0$, $a \cdot b = \gamma_1 \partial(b_0 a)$, $A \cong C[x, y]/(x^2, y^2, xy)$.

The vector space $(a, b_0 a)$ is the unique maximal ideal of A. For $u \in (a, b_0 a)$, $w \in B/A\partial(A)$, $w_0 u \in (a, b_0 a)$ and $u \cdot w = 0$.

If $\gamma_0 \neq 0$, we have $a * a = \gamma_0 b_0 a$, $a * a * a = 0$, $(b_0 a) \cdot b = \frac{3}{4} \gamma_0 \partial(b_0 a)$, $A \cong C[x]/(x^3)$.

The vector space (a) is a unique maximal ideal of A.

For $u \in (a)$, $w \in B/A\partial(A)$, $w_0 u \in (a)$ and $u \cdot w = 0$.

For $\beta, \gamma_0, \gamma_1, \chi \in C$.

Sketch proof of theorem 2: From an earlier lemma we have $b_0 \partial(b_0 a) = c_0 \partial(a) + c_1 \partial(b_0 a)$ for $c_0, c_1 \in C$. If we set $c_0, c_1 = 0$, we have the following relations:

$\beta = 0$, $a * (b_0 a) = 0$, $a * a = (\gamma_1 + 1)\chi 1_A + \gamma_0 b_0 a$, $a \cdot b = \gamma_0 \partial(a) + \gamma_1 \partial(b_0 a)$, $a \cdot \partial(a) = \frac{1}{2} \gamma_0 \partial(b_0 a)$, and $\gamma_0 \chi = 0$.

We consider when $\gamma_0 = 0$. Recall from definition 3, that for $u, v \in B$, $a' \in A$, we have

$u_0(a' \cdot v) - a' \cdot (u_0 v) = (u_0 a') \cdot v$.

When we set $u = v = b$ and $a' = a$, we have $b_0(a \cdot b) - a \cdot (b_0 b) = (b_0 a) \cdot b$. Through several calculations we find that $\chi = 0$ and $a * a = 0$. Hence $A \cong C[x, y]/(x^2, y^2, xy)$.

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