On certain algebraic structures associated with Lie (super)algebras

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Abstract:

This talk is to provide a certain correspondence with an algebraic and a geometric structures with respect the Lie algebras

Introduction

This note is to deal with a cetain survey based on our papers mainly and to give several examples of triple systems, furthermore to describe a history of Jordan rivers in nonassociative algebras from the author's viewpoint. On the other hand, this work with respect to Jordan and Lie structures is in close contact with a symmetric (super)space equipped with commplex structure, since the

tangent space of the symmetric (super)space is a δ-Lie triple system ($\delta = \pm 1$).
From mathematical history's viewpoint, the concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$.
Nonassociative algebras are rich in algebraic

structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry (for example, [11], [46], [59], [61], [64]). Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([5], [10]-[13], [15], [22], [26], [28], [29], [33], [34],

[35], [45], [54], [55], [60], [65]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [6]-[8], [20], [23], [24], [40]-[45], [50]-[53], [56]-[58]) by using the standard embedding method ([22], [48], [49], [57], [63]). In particular, the generalized Jordan triple system of second order, or $(-1,1)$-Freudenthal Kantor triple system (for short $(-1,1)$-FKTS), is a useful concept ([13]-[21], [41]-[44], [47], [62]) for the

constructions of simple Lie algebras, while the $(-1,-1)$-FKTS plays the same role ([6], [22], [25], [27], [30], [32], [33]) for the construction of Lie superalgebras, while the δ-Jordan Lie triple systems act similarly for that of Jordan superalgebras ([23], [24], [56]). Specially, we have constructed a model of basic Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ ([22], [25], [27]).

As a final comment of this introduction, we provide well-known results due to O. Loos as follows; if A is a unital commutative Jordan algebra with a unital element e, that is, satisfying $(xy)x^2 = x(y(x^2))$, and $xy = yx$, then the triple product given by

$$\{xyz\} = (xy)z + x(yz) - y(xz)$$

defines a Jordan triple system with $\{xey\} = xy$, i.e., it satisfies the two relations $\{xy\{abc\}\} =$ $\{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$ (this relation is often called a fundamental identity), and $\{xyz\} = \{zyx\}$ (this relation is called a commutative identity, since $xy = \{xey\} = \{yex\} = yx$) and next the new triple product $[xyz]$ given by

$$[xyz] = \{xyz\} - \{yxz\}$$

defines a Lie triple system.

Briefly summarizing this article, we will generalize these results and exhibit examples of Lie (super)algebras associated with generalized Jordan triple systems. Toward to

its applications, in particular, we will give a construction of symmmetric (super)spaces with an almost complex structure (i.e., eqipped with Nijenhuis operator).
Roughly describing, we have an illustration for our concept ;

Algebraic structures $\Longleftrightarrow$
Geometric structures.

For examples, it seems that there are certain algebraic structures associated with symmetric, R-symmetric, homogeneous spaces, totally geodesic manifold, and symmetric domains, etc.

1 Definitions and Results

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order) ([41]-[45]) .

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \tag{2}$$

where $L(a,b)c := (abc)$ and $K(a,b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies

(1) and the following condition

$$(abc) = (cba), \ i.e., \ K(a,c)b = 0. \qquad (3)$$

The JTS is a special case in the GJTS of 2nd order since $K(x,y) \equiv 0$.
We next can generalize the concept of GJTS of 2nd order as follows (see [13], [14], [18], [22], [28], [36] [63] and the earlier references therein).
For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that

satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{4}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0, \tag{5}$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca), \tag{6}$$

is called an

$(\varepsilon,\delta)-Freudenthal-Kantor\ triple\ system$

(for short (ε, δ)-FKTS). An (ε, δ)-FKTS is said to be *unitary* if $Id \in \{K(a, b)\}_{span}$.
A triple system satisfying only the identity (4) is called a *generalized* FKTS (for short GFKTS), while the identity (5) is called the *second order condition* (this condition needs to construct of 5-graded Lie (super)algebras).

Remark From the relation Eq. (6), we note that

$$K(b, a) = -\delta K(a, b). \tag{7}$$

A triple system is called a (α, β, γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha < x, y > z + \beta < y, z > x + \gamma < z, x > y,$$

where $< x, y >$ is a bilinear form such that $< x, y >= \kappa < y, x >$, $\kappa = \pm 1$, $\alpha, \beta, \gamma \in \Phi$. From now on we will mainly consider this type of triple system.

An (ε, δ)-FKTS is said to be *balanced* if there is a bilinear form $< x, y > \in \Phi^*$ such that

$K(x, y) =< x, y > Id$, that is,
$dim\ \{K(x, y)\}_{span} = 1$ holds.

Remark We note that a balanced triple system (i.e., it fulfills $K(x, y) =< x, y > Id$) is unitary, since $Id \in \{K(x, y)\}_{span}$.

Triple products are denoted by (xyz), $\{xyz\}$, $[xyz]$ and $< xyz >$ upon their suitability.

Remark We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$-FKTS. Thus we can construct the corresponding Lie algebras by means of the

standard embedding method ([6], [13]-[19], [21], [22], [25], [27], [43]).
For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc], a, b, c \in V$ is called a δ-*Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{gathered}[abc] = -\delta[bac], \\ [abc] + [bca] + [cab] = 0, \\ [ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]],\end{gathered} \tag{8}$$

where $a, b, x, y, z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS*, by ([14]). Note that the set $L(V, V)$ of all left multiplications $L(x, y)$ of V is a Lie subalgebra of $Der\ V$, where we denote by $L(x, y)z = [xyz]$.

Proposition 1.1 ([13]-[16], [22]) *Let* $(U(\varepsilon, \delta), < xyz >)$ *be an* (ε, δ)*-FKTS. If* J *is an endomorphism of* $U(\varepsilon, \delta)$ *such that* $J < xyz >=< JxJyJz >$ *and* $J^2 = -\varepsilon\delta Id$, *then* $(U(\varepsilon, \delta), [xyz])$ *is a LTS (if* $\delta = 1$*) or an* *anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] :=$$

$$< xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > .$$

Remark *Note that for the case of $\varepsilon = -1, \delta = 1$ and $K(x, y) = 0$, we have a special case in Prop.1.1, that is, it implies that $J = Id$, $\{xyz\}$ is the JTS and $[xyz] = \{xyz\} - \{yxz\}$ is the LTS described in Introduction.*

Corollary *([13]) Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] =$$

$$\begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$),
$L(U(\varepsilon, \delta)) = D(T(\varepsilon, \delta), T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta)$,
associated with $T(\varepsilon, \delta)$ where
$D(T(\varepsilon, \delta), T(\varepsilon, \delta))$ is the set of inner derivations of $T(\varepsilon, \delta)$;

$$D(T(\varepsilon, \delta), T(\varepsilon, \delta)) := \left\{ \begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{pmatrix} \right\}$$

$$T(\varepsilon, \delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U(\varepsilon, \delta) \right\}_{span}.$$

We use the following notation:

$$\mathbf{k} := \{K(x, y) \in End\, U(\varepsilon, \delta) | x, y \in U(\varepsilon, \delta)\} \; and$$

$$\{EFG\} := EFG + GFE, \; \forall E, F, G \in \mathbf{k}.$$

Then, we may make the structure of a JTS **k** with respect to the triple product $\{EFG\} \in \mathbf{k}$, hence $[EFG] = \{EFG\} - \{FEG\}$ has a structure of LTS ([20]).

We next introduce an analogue of Nijenhuis tentor in differential geometry defined by

$$N(X,Y) =$$

$$[JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y],$$

$$\forall X, Y \in T(\varepsilon, \delta)$$

and $J = \begin{pmatrix} 0 & \varepsilon \\ -\delta & 0 \end{pmatrix}$, that is $J^2 = -\varepsilon\delta Id$, hence if $J^2 = -Id$, then this (the case of $\varepsilon\delta = 1$) has a structure of allmost complex.

Proposition 1.2 *Let U be a* (ε, δ)*-FKTS,* $T(\varepsilon, \delta)$ *be the* δ*-LTS and* $L(U)$ *be the standard embedding Lie (super)algebra associated with U. Then the following are equivalent:*

(i) $N(X, Y) = 0, \forall X, Y \in T(\varepsilon, \delta)$*,*

(ii)

$\varepsilon\delta L(y, x) - \varepsilon L(x, y) = K(x, y), \ \forall x, y \in U(\varepsilon, \delta)$.

This $J \in End\ T(\varepsilon, \delta)$ may generalize on $\tilde{J} \in End\ L(U)$ defined by

$$\tilde{J} := JD(X, Y)J^{-1} \oplus JZ, \ \ \forall X, Y, Z \in T(\varepsilon, \delta).$$

Then we note that $\tilde{J}$ has an interesting property, for example, an automorphism of $L(U)$ associated with U.

Proposition 1.3 *For a (ε, δ)-FKTS U and $L(U)$ as in above Proposition, assuming $\varepsilon = \delta$ and $K(x, y) = L(y, x) - \varepsilon L(x, y)$, then the elements $f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in sl(2)$ (i.e., $[f, g] = h,\ [f, h] = -2f,\ [g, h] = 2g$) are* *derivations of $L(U)$.*

Remark We note that
$L(U) = L(U(\varepsilon,\delta)) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$
is the five graded Lie (super)algebra such that
$U(\varepsilon,\delta) \oplus U(\varepsilon,\delta) = L_{-1} \oplus L_1 = T(\varepsilon,\delta)$
(δ-LTS), $L_{-2} = \mathbf{k}$ (JTS) and
$D(T(\varepsilon,\delta), T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$ (the
derivation of $T(\varepsilon,\delta)$) equipped with
$[L_i, L_j] \subseteq L_{i+j}$ and
$L_{-1} \oplus L_1 = L(U)/L_{-2} \oplus L_0 \oplus L_2$. In
Introduction, we had used the notation

$g = g_{-1} \oplus g_0 \oplus g_1$ instead of $L_{-1} \oplus L_0 \oplus L_1$. This Lie (super)algebra construction is one of reasons to study nonassociative algebras and triple systems without using root systems (for a Lie superalgebra, refer to ([9], [12], [60])). Also this construction can be represented by the concept of a normal triality algebra (see [34], [35]).

2 Examples of (ε, δ)-JTS

We will consider here examples of the special case defined by bilinear forms $< x, y >$, that is, an (ε, δ)-JTS of (α, β, γ) triple systems equipped with $K(x, y) \equiv 0$. Moreover, we give two examples (Prop. 2.2 and Prop.2.3) without the cases of (ε, δ)-JTS.

Example 2.1 Let V be a vector space with a symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z+ < y, z > x- < z, x > y$$

defines on V a $(-1, 1)$-JTS.
Note that $(-1, 1)$-JTS is same as the JTS.
Example 2.2 Let V be a vector space with an anti-symmetric bilinear form $< x, y >$.
Then

$$< xyz >=< x, y > z+ < y, z > x- < z, x > y$$

defines on V a $(1, -1)$-JTS.
Example 2.3 Let V be a vector space with a

symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z- < y, z > x$$

defines on V a $(-1, -1)$-JTS.

Example 2.4 Let V be a vector space with an anti-symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z- < y, z > x$$

defines on V a $(1, 1)$-JTS.

Example 2.5 Let V be a set of alternative matrix $Asym(n, \Phi) = \{x | {}^t x = -x\}$, where ${}^t x$ denote the transpose matrix of x. Then

$$< xyz >= x^t yz - \varepsilon z^t yx, \quad where \; \forall \; x, y, z \in V$$

defines on V a $(\varepsilon, -\varepsilon)$ JTS, that is, the case of $\varepsilon = -1 \Rightarrow$ JTS.

Remark Let V be the set of $p \times q$ matrix $Mat(p, q; \Phi)$. Then this vector space V is a JTS with respect to the product $\{xyz\} = x^t yz + z^t yx, \; \forall x, y, z \in V$.

Proposition 2.1 *Let* $(U, < xyz >)$ *be an* (ε, δ)*-JTS. Then the triple system is a* δ*-LTS with respect to the new product*

$$[xyz] =< xyz > -\delta < yxz > . \qquad (9)$$

In the next section 3 subsection we study the case of an (ε, δ)-FKTS, but we give first two examples which are not (ε, δ)-JTS as it follows.

Proposition 2.2 *Let* $(U, < xyz >)$ *be a triple system with* $< xyz >=< y, z > x$ *and*

$< x, y > = -\varepsilon < y, x >$. *Then this triple system is an* (ε, δ)*-FKTS.*

Proposition 2.3 ([16], [18]) *Let U be a balanced* $(1,1)$*-FKTS satisfying* $<< xxx >, x > \equiv 0$ *(identically) and* $< x, y >$ *is nondegenerate. Then U has a triple product defined by*

$$< xyz > = \frac{1}{2}(< y, x > z + < y, z > x + < x, z > y). \tag{10}$$

Note that the balanced $(1,1)$-FKTS induced from an exceptional Jordan algebra is closely related to the 56 dimensional meta symplectic geometry due to H. Freudenthal ([13], [15], [16], [18] and the earlier references therein). Also the correspondence of a quaternionic symmetric space and the balanced (1,1) FKTS has been studied in ([5]). On the other hand, for $(-1,-1)$ -FKTS, see ([6] and[7], [30], [31]).

3 Examples of Lie (super)algebras associated with (ε, δ) FKTS

We will exhibit the examples of some triple systems and Lie (super)algebras associated with their triple systems.

Example a) $C(n+1)$ type is of dimension $\dim C(n+1) = 2n^2 + 5n + 1$.

Let U be the set of matrices $M(1, 2n; \Phi)$.

Then, by Example 2.2, it follows that the triple product

$$L(x,y)z =< xyz >:$$

$$=< x,y > z+ < y,z > x- < z,x > y$$

such that the bilinear form fulfills $< x,y >= - < y,x >$, is a $(1,-1)$-JTS, since $K(x,y) \equiv 0$ (identically). Furthermore, the standard embedding Lie superalgebra is 3-graded and of $C(n+1)$ type. For the extended Dynkin diagram, we obtain

$$L_{-1} \oplus L_0 \oplus L_1 :=$$

$$\left\{ \begin{pmatrix} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{pmatrix} \middle| \varepsilon = 1 = -\delta \right\}_{span} \oplus \left\{ \begin{pmatrix} e \\ f \end{pmatrix} \right\}_{s}$$

$$\otimes\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \qquad\qquad \alpha_n \quad \alpha_{n+1}$$

$$\| \quad > \quad \circ - \circ - - - - - \circ <= \circ$$

$$\otimes\ \alpha_0$$

$$= C(n+1)\ type\ (\alpha_1\ \otimes\ deleted).$$

Also, we obtain

$$L_0 := \left\{ \begin{pmatrix} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{pmatrix} \middle| \varepsilon = 1 = -\delta \right\}_{span} \cong$$

$$\begin{matrix} \alpha_2 & & \alpha_3 & & & & & & & & \alpha_n & & \alpha_{n+1} \\ \circ & - & \circ & - & - & - & - & - & - & - & \circ & <= & \circ \end{matrix}$$

$$= C_n \oplus \Phi Id \; (\alpha_1 \; \otimes \; and \; \alpha_0 \; \otimes \; deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $C(n+1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \otimes$.

Example b) $B(n,1)$ and $D(n,1)$ type are of dimension $\dim B(n,1) = 2n^2+5n+5$ and $\dim D(n,1) = 2n^2+3n+3$, respectively.
Let U be the set of matrices $M(1,l;\Phi)$. Then, by straihtfoward calculations, it follows that the triple product

$$L(x,y)z =< xyz >:$$

$$= \frac{1}{2}(< x, y > z- < y, z > x+ < z, x > y)$$

such that the bilinear form fulfills

$< x, y > = < y, x >$ is a $(-1,-1)$-FKTS.
Furthermore, the standard embedding Lie superalgebra is 5-graded and of $B(n, 1)$ type if $l = 2n + 1$, or of $D(n, 1)$ type if $l = 2n$. For the extended Dynkin diagram, we obtain from the results of § 1 the following.
For the case of $B(n, 1)$ type we have

$$L_{-2} \oplus L_0 \oplus L_2 := D(T(-1,-1), T(-1,-1)) =$$

$$\left\{ \begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix} \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\alpha_0 \quad \alpha_1 \quad \alpha_2 \qquad\qquad\qquad \alpha_n \quad \alpha_{n+1}$$

$$\circ => \otimes - \circ - - - - - \circ => \circ$$

$$= A_1 \oplus B_n \ type \ (\alpha_1 \ \otimes \ deleted).$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\alpha_2 \quad \alpha_3 \qquad\qquad\qquad \alpha_n \quad \alpha_{n+1}$$

$$\circ - \circ - - - - - \circ => \circ$$

$$= B_n \oplus \Phi Id \ (\alpha_1 \ \otimes \ and \ \alpha_0 \ \circ \ deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $B(n,1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \circ$.
Similarly, for the case of $D(n,1)$ type we have $L_{-2} \oplus L_0 \oplus L_2 \cong A_1 \oplus D_n, \ L_0 \cong D_n \oplus \Phi Id.$
We note that this triple system is balanced and with a complex structure of Nijenhuis tensor zero, since
$K(x,y) =< x,y > Id = L(x,y) + L(y,x)$

Remark We note that the case of balanced is discussed in ([18], [28]). On the other hand, for the construction of simple exceptional Lie algebras G_2, F_4, E_6, E_7, E_8, refer to ([16], [18], [21]). Also, for the construction of simple Lie superalgebras $G(3)$, $F(4)$, $D(2,1,\alpha), P(n), Q(n), H(n)$, $S(n)$ and $W(n)$, refer to ([22], [25], [27], [31]). Of course, these construction are created from the concept of triple systems without using systems of roots. Thus, moreover, these examples imply that our methods may apply the symmetric superspace (the case of $\delta = -1$) as well as the structures (see, [5], [46]) of the symmetric spaces (the case of $\delta = 1$), however we will not go into the details and in future, we will discuss it.

In the rest of this section, we will consider the constructions of simple B_3-type Lie algebra associated with several triple systems (the case of $\varepsilon = -1$ and $\delta = 1$), more easily. That is, we will give several examples; (c) the case of a JTS (i.e., $(-1,1)$-FKTS with $K(x,y) \equiv 0$), (d) the case of a GJTS of 2nd order (i.e., $(-1,1)$-FKTS with $dim\{K(x,y)\}_{span} = 1$), (e) the case of a GJTS of 2nd order (i.e, $(-1,1)$-FKTS with $dim\{K(x,y)\}_{span} = 3$).

Example c) We study the case of $g_{-1} = U = Mat(1,5;\Phi)$. Hereafter in this subsection, as a reason of traditional notation, we often would like to denote by g_i instead of $L_i, (i = 0, \pm 1, \pm 2)$ and by $\{xyz\}$ instead of

$< xyz >$.
In this case, g_{-1} is a JTS with respect to the product

$$\{xyz\} = x^{t}yz + y^{t}zx - z^{t}xy, \ \forall \ x, y, z \in g_{-1}$$

where ${}^{t}x$ denotes the transpose matrix of x.
By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_3-type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and a LTS

$T(U) = g_{-1} \oplus g_1$. Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong B_2 \oplus \Phi H,$$

$$where\ H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Here in view of the relations
$[S(x,y), L(a,b)] = L(S(x,y)a,b)+$
$L(a, S(x,y)b)$, and
$[A(x,y), L(a,b)] = L(A(x,y)a,b)$
$-L(a, A(x,y)b)$ for all $L(a,b) \in End\ U$, when

$\varepsilon = -1, \delta = 1$, we use the following notations;

$$Der\ U := \{L(x, y) - L(y, x)\}_{span},$$

$$Anti - Der\ U := \{L(x, y) + L(y, x)\}_{span},$$

$$g_0 = \left\{ \begin{pmatrix} L(x, y) & 0 \\ 0 & -L(y, x) \end{pmatrix} \right\}_{span}$$

$$= \{S(x, y) + A(x, y)\Big\}_{span}$$

where $S(x, y) := L(x, y) - L(y, x) \in$ $Der\ U,\ A(x, y) := L(x, y) + L(y, x)$ $\in Anti - Der\ U$, this case is $\varepsilon = -1,\ \delta = 1$.
Example d) Second, we study the case of $g_{-1} = U = Mat(2, 3; \Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1, 1)$-FKTS) with $dim\ \{K(x, y)\}_{span} = 1$ with respect to the product

$$\{xyz\} = x^t yz + z^t yx - z^t xy,\ \forall\ x, y, z \in g_{-1}.$$

By straightforward calculations, it can be shown that the standard embedding Lie algebra $L(U) = g$ is a 5-graded B_3-type Lie algebra with $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and $dim\ g_{-2} = dim\ g_2 = dim\ \{K(x, y)\}_{span} = 1.$ Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong A_1 \oplus A_1 \oplus \Phi H,$$

$$where\ H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Furthermore, we obtain a LTS $T(U)$ of

$$dim\ T(U) = dim\ (g_{-1} \oplus g_1) = 12,$$

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_1 \oplus A_1 \oplus A_1 \cong Der$$

Also, in this case, we note that
$T(U) = L(U)/Der\ T(U) =$
$g/(g_{-2} \oplus g_0 \oplus g_2)(= g_{-1} \oplus g_1)$ is the tangent space of a quaternion symmetric space of dimension 12, since $T(U)$ is a Lie triple system associated with g_{-1}.

Example e) Third, we study the case of $g_{-1} = U = Mat(1, 3; \Phi)$. In this case, g_{-1} is a

GJTS of 2nd order (i.e., $(-1,1)$-FKTS) with respect to the product

$$\{xyz\} = x^t yz + z^t yx - y^t xz,$$

$$K(x,y)z = \{xzy\} - \{yzx\}, \ \forall \ x, y, z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 5-graded B_3-type Lie algebra with $g = g_{-2} \oplus \cdots \oplus g_2$ and $dim \ g_{-2} = dim \ g_2 = 3$. Thus, we have

$$g_{-2} = \{K(x,y)\}_{span} = Alt(3,3;\Phi).$$

Furthermore, we obtain a LTS $T(U)$ of $dim\ T(U) = dim\ (g_{-1} \oplus g_1) = 6$,

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_3 \cong Der\ T(U).$$

This case $g_{-2} = \{K(x,y)\}_{span} = \mathbf{k}$ has the structure of a JTS (cf. section 2).

Remark We remark that the cases (a)and (b) (resp. (c),(d),(e)) are $\delta = -1$ (resp. $\delta = 1$).

Remark For the root system $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$and the highest root $-\rho = \{\alpha_1 + 2\alpha_2 + 2\alpha_3\}$ of the simple Lie algebra B_3, the case of (c) means that $g_{-1} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$ and $g_{-2} = \{0\}$, the case of (d) means that $g_{-1} = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3\}$ and $g_{-2} = \{-\rho\}$, the case of (e) means that

$g_{-1} = \{\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3\}$ and
$g_{-2} = \{\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$.

4 Mathematical physics Remarks

In this section, we give several references of mathematical physics in our works.
We note that there are applications toward the Yang-Baxter equations associated with triple systems ([26], [39], [57]) and also toward the field theory associated with Hermitian triple systems ([37], [38]). For other mathematical physics, it seems that the books ([28], [33]) are useful.

5 History from a certain personal viewpoint

For a mathematical history, in particular for Jordan rivers, we describe belows:
This brief history (with respect to nonassociative algebras) is a story from author's personal aspect (judgement). Triple systems (ternary algebras) have first been appeared from Prof. N. Jacobson and

continued by Profs. O. Loos, K. Meyberg and E. Neher of students of Prof. M. Koecher in Germany, also certain triple systems associated with the geometry of 56 dimensional due to Prof. Freudenthal have been studied by Prof. J. Faulkner (resp. K. Meyberg) of the student of Prof. N. Jacobson (resp. Prof. M. Koecher).

On the other hand, there is a history;

H. Freudenthal (Netherlands) $--$ >K. Yamaguti (Japan) or I. L. Kantor (Russian and Sweden, he was born in Belarus) −− > Author (N. Kamiya) −− > D. Mondoc (but these arrows are no students), however, Dr. Mondoc is only a student of Prof. Kantor in Sweden.

Profs. O. Loos and E. Neher in the student of Prof. M. Koecher in Germany are working in Jordan triple systems and Jordan pairs. Profs. Kantor, Yamaguti, S. Okubo and author(N. Kamiya) are studying in their generalizations, for example, refer to N. Kamiya and S. Okubo

"Representation of (α, β, γ) *triple systems,"*, Linear and Multilinear Algebras, **58** no.5-6 (2010) 617-643. This history is a story whithout using concept of root systems and Cartan matrix in Lie algebras, in particular, is a study for triple systems.
Note that there are a lot of mathematician in nonassociative algebras (for Lie algebras), but a little groups in triple systems or Jordan algebras. For example, Profs. E. Zelmanov, K. McCrimmon, B. Allison, V. Kac, I.

Shestakov, H. Petersson, M. Racine, H. Asano, I. Satake, M. C. Myung, A. Elduque, C. Martinez, S. Gonz̀alez, S. Okubo and author, may be, only a few. Furthermore in addition, the book *"A Taste of Jordan Algebras"* (Springer, 2003) written by Prof. K. McCrimmon of a student in N. Jacobson is described about a history of the Jordan river. It here emphasize that this historical survey of certain Jordan algebras until the end of the 20th century and the beginning of 21th century is my (author) aspect (viewpoint). In addition to above river, for a certain example, for our imaginative illustrations with respect to a generalization of numbers;

$$(\sharp)\ \mathbf{R} \to \mathbf{C} \to \mathbf{H} \to \mathbf{O}(octonion)$$

$$\to \mathbf{H_3}(\mathbf{O})(Jordan\ algebra\ of\ 27\ dim) \to$$

$$\mathbf{M}(\mathbf{H_3}(\mathbf{O}))(metasymplectic\, geometry\, of\, 56\, dim) \to$$

$$\mathbf{T}(\mathbf{H_3}(\mathbf{O}))(symmetric\ space\ of\ 112\ dim) \to$$

$\mathbf{E_8}$(*exceptional simple Lie algebra of* 248 *dim*).

On the other hand, there is other river also,

$$(\sharp\sharp)\ \mathbf{O} \to \mathbf{C} \otimes \mathbf{O},\ \mathbf{H} \otimes \mathbf{O}\ and\ \mathbf{O} \otimes \mathbf{O}$$

$$(Freudenthal's\ magic\ square) \to$$

$$\mathbf{T}(\mathbf{O} \otimes \mathbf{O})(symmetric\ space\ of\, 128\ dim)$$

$$\to \mathbf{E_8}.$$

For another way, there is a river of Prof. Tits (called Tits's construction) as follows.

($\sharp\sharp\sharp$) The case $A = \mathfrak{A}_0 \otimes \mathfrak{J}_0$; (with $dim\ A = 7 \times 26,\ dim\ Der\ (A) = 66$), where the base field Φ is an algebraically closed field of characteristic 0.

$L(A) = Der(A) \oplus A \cong E_8,\ Der(A) = Der\mathfrak{A} \oplus Der\mathfrak{J} \cong G_2 \oplus F_4 =< D(X,Y) >_{span}$. Here $\mathfrak{A}_0$ denote $\{x \in \mathbf{O} | trace\ x = 0\}$ and $\mathfrak{J}_0 = \{x \in H_3(\mathbf{O}) | Trace\ x = 0\}$. For the product of A, $X \circ Y = (a * b) \otimes (x * y)$ and with respect to the Lie product of $L(A)$, $[X,Y] = D(X,Y) + X \circ Y$, then the vector space $(A, \ \circ)$ has an algebraic structure of satisfying

$$D(X \circ Y, Z) + D(Y \circ Z, X) + D(Z \circ X, Y) = 0,$$

where $X, Y, Z \in A$ ([19] and see the earlier references therin).

If we set $\mathfrak{J} = H_3(\mathbf{O}) \to H_3(\mathfrak{A})$ $(= B)$, then we have the following table;

$\backslash$	$dim B = 6$	$dim B = 9$	$dim B = 15$	$dim B = 27$
$dim \mathfrak{A} = 1$	A_1	A_2	C_3	F_4
$dim \mathfrak{A} = 2$	A_2	$A_2 \oplus A_2$	A_5	E_6
$dim \mathfrak{A} = 4$	C_3	A_5	D_6	E_7
$dim \mathfrak{A} = 8$	F_4	E_6	E_7	E_8

Here note that $L(A)/(G_2 \oplus F_4)$ is a reductive homogeneous space with 182 dimension.
It seems that there are several researchers group's tradition for these study and furthermore, for a nonassociative world of 21th century, Spanish, Portuguese and middle Europe scholars groups will glow with respect to the study (may be, Prof. Elduque's group mainly).
For algebraic structures of nonassociative subject (AMS classification 17) related with

geometry, about 20th century, roughly speaking, we may describe as follows, for example (in my opinion);

Jordan algebras researchers (E. Artin origin),

Lie algebras researchers (N. Jacobson origin).

In summarizing about Jordan algebras or triple systems, we have the following diagrams (a generalization of complex and quaternionic numbers):

octonion, pseudo octonion algebras and triple systems $\Longrightarrow$

Jordan algebras +Lie (super)algebras +symmetric composition algebras

$\Longrightarrow$**mathematical algebras** (author's new phrase)

In final comments (although they had described in the introduction), also we emphasize that nonassociative algebras are rich in algebraic structures, and they provide important common ground for various branches of mathematics, not only pure algebra and mathematical physics (for example, Pierce decompositions, Yang-Baxter equations and quark theory), but also analysis (Jordan C^* algebras or JB^* triple), topology (racks or quandles), and geometries (generalized symmetric spaces, convex cones or bounded symmetric domaines, in particular). Hence, in future aspect, it seems that the triple systems (or ternary product) without using unit elements are useful concept for several subjects of sciences as well as the situation of symmetric spaces.

For a recent book of Springer Pub. (Proc. Math.and Stastics, vol. 427, (2023)), it seems that there is a lot of study in nonassociative world.

6 Geometric structures

6.1 A generalized curvature and torsion tensors

Let $L = L(U(\varepsilon, \delta)) = L(W, W) \oplus W$ be the Lie algebra defined from a δ LTS as in the section one, that is, the δ-LTS $W = T(\varepsilon, \delta) = L_{-1} \oplus L_1$ is induced from $L_{-1} = U(\varepsilon, \delta)$ (as L_{-1} has the structure of a (ε, δ)-FKTS).

We now introduce a generalization of

covariant derivative $\bigtriangledown$ in differential geometry as follows; $\bigtriangledown : L \to \text{End } L$ defined by

$\bigtriangledown_X Y = [X, Y] = -\delta[Y, X]$,

$\bigtriangledown_X [Y, Z] = [YZX] = -\delta[ZYX]$,

$\bigtriangledown_{[X,Y]} Z = -[XYZ] = -\delta[YXZ]$,

$\bigtriangledown_{[X,Y]} [V, Z] = [[V, Z][X, Y]] = -\delta[[X, Y][V, Z]]$,

for any $X, Y, Z \in W$.

Furtheremore, a generalized curvature tensor defined by

$$C_\delta(X,Y) = \nabla_X \nabla_Y - \delta \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad (11)$$

is identically zero, i.e., $C_\delta(X,Y) = 0$ in L, for any $X, Y \in W$. Indeed, we demonstrate the proof below.
First we calculate

$$C_\delta(X,Y)Z = (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)Z - \nabla_{[X,Y]}Z$$

$$= \nabla_X [Y,Z] - \delta \nabla_Y [X,Z] + [XYZ]$$

$$= [YZX] - \delta[XZY] + [XYZ]$$

$$= [YZX] + [ZXY] + [XYZ] = 0.$$

Second, it follow

$$C_\delta(X,Y)[V,Z] =$$

$$(\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)[V,Z] - \nabla_{[X,Y]}[V,Z]$$

$$= [X,[VZY]] - \delta[Y,[VZX]] + \delta[[X,Y],[V,Z]] =$$

$$[X, L(V,Z)Y] - \delta[Y, L(V,Z)X] - L(V,Z)[X,Y] = 0$$

(by $[Y, L(V,Z)X] = -\delta[L(V,Z)X, Y]$ and $[[X,Y],[V,Z]] = -\delta[[V,Z],[X,Y]]$)
for any $X, Y, Z, V \in T(\varepsilon, \delta)$.
However a generalized torsion tensor defined by

$$S_\delta(X,Y) = \bigtriangledown_X Y - \delta \bigtriangledown_Y X - [X,Y] \quad (12)$$

is not zero, since it gives
$S_\delta(X,Y) = [X,Y] - \delta[Y,X] - [X,Y] = [X,Y]$.
In final comments of this section, for δ-LTS $W = T(\varepsilon, \delta)$, we recall the Nijenhuis operator

in the section one;

$$N(X,Y) =$$

$$[JX,JY] + J^2[X,Y] - J[JX,Y] - J[X,JY],$$

where J is an almost complex structure on W, this concept (the case of $\delta = 1$) is appeared in ([36]), hence we may consider a generalization with respect to the super symmetric space (the case of $\delta = -1$).

If we set $J = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, or $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then we have $J^2 = -Id,\ or\ J^2 = Id$ respectively, and it seems that there is a twisted or a straight (para complex) property in the sence of W. Bertram.

6.2 magic square table of exceptional simple Lie algebras

Following ([34],[35]), we consider the simple Lie algebras associated with normal triality algebras A, that is, the construction of 5-graded exceptional Lie algebras $L(A) = g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and $g_{-1} = A$. Here we denote that the base field is an algebraically closed field F of characteristic 0.

I) $A = \mathbf{O} \otimes \mathbf{O}$ (tensor product case, $Der\ \mathbf{O} \cong G_2$, $dim\ A = 64$, $dim\ g_{-2} = dim\ g_2 = 14$).

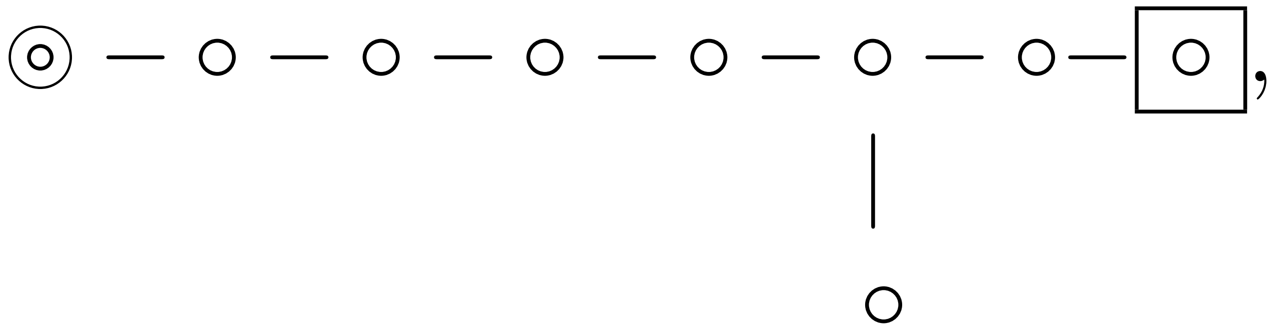
For subalgebras of A, if we use the notation of $A = A_1 \otimes A_2,\ dim\ A_1, dim\ A_2$, then the Lie algebras obtained from their subalgebras are following:

$$L(A) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \cong E_8,$$

$$g_{-2} \oplus g_0 \oplus g_2 \cong D_8, A = g_{-1}$$

$dimA_2 \backslash dimA_1$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

For this case's E_8, considering the Extended Dynkin diagram, we have
$g_{-1} \oplus g_1 \cong L(A)/(g_{-2} \oplus g_0 \oplus g_2) = E_8/D_8$
with dim $(g_{-1} \oplus g_1)$=128;



$\boxed{\circ}$ omitted $\cong D_8$, and $\circledcirc$ is highest root.

II) $A = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ (balanced case, $dim\ A = 56$, $dim\ g_{-2} = dim\ g_2 = 1$), where $\mathbf{a},\ \mathbf{b} \in H_3(\mathbf{O})$ (exceptional Jordan algebra with 27 dimension) and $\alpha,\ \beta \in F$. The Lie algebra constructed from this algebra A is the

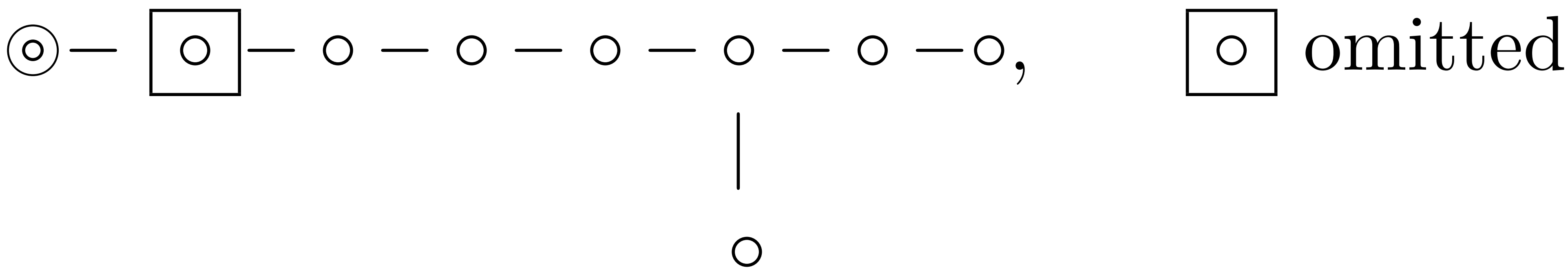
following.
$L(A) \cong E_8,\ \ g_{-2} \oplus g_0 \oplus g_2 \cong E_7 \oplus A_1,$
$g_0 \cong E_7 \oplus gl(1), A = g_{-1}.$
To change the notation
$H_3(\mathbf{O}) \to H_3(\mathfrak{A})(= B)$, here $\mathfrak{A}$ is a Hurwitz algebras over F.
$\forall \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in \begin{pmatrix} F & B \\ B & F \end{pmatrix} = A$, with respect to the $dimB$, Lie algebras $L(A)$ obtained from B are the following.

$dim B$	1	6	9	15	27
$dim A$	4	14	20	32	56
$dim L(A)$	14	52	78	133	248
$L(A)$	G_2	F_4	E_6	E_7	E_8

For this case's E_8, considering the Extended Dynkin diagram, we have

$g_{-1} \oplus g_1 \cong L(A)/(g_{-2} \oplus g_0 \oplus g_2) = E_8/(A_1 \oplus E_7)$

with dim $g_{-1} \oplus g_1$=112;



$\cong\ A_1 \oplus E_7.$

Remark This type construction of type II with $dim\ A = 56$ has been first studied by H. Freudenthal (called a metasymmetric geometry equipped $P \times Q$ and $\{P, Q\}$'s notations). And this concept is characterized by a triple system (or a ternary algebra) called a generalized Zorn's vector matrix

([13]-[16],[18],[35] the references of therein).

6.3 bisymmetric spaces associated with exceptional simple Lie algebras

Following the books due to O. Loos or W. Bertram with respect to symmetric spaces, we have associated to a symmetric space $M = G/H$ a Lie triple system T (as the tangent space of the symmetric space is a Lie triple system).

We consider a concept of bisymmetric space $(B_\alpha, B_\beta, B_\gamma, B_\delta)$ in Lie triple systems pair defined as follows:

$(I)\ dim\ B_\delta/dim\ B_\gamma = dim\ B_\gamma/dim\ B_\beta =$

$$dim\ B_\beta/dim\ B_\alpha = 2,\ and$$

$$B_\alpha < B_\beta < B_\gamma < B_\delta$$

as Lie triple subsystem's series of the Lie triple system $g_{-1} \oplus g_1$ of 5-graded Lie algebra $g = L(A) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$

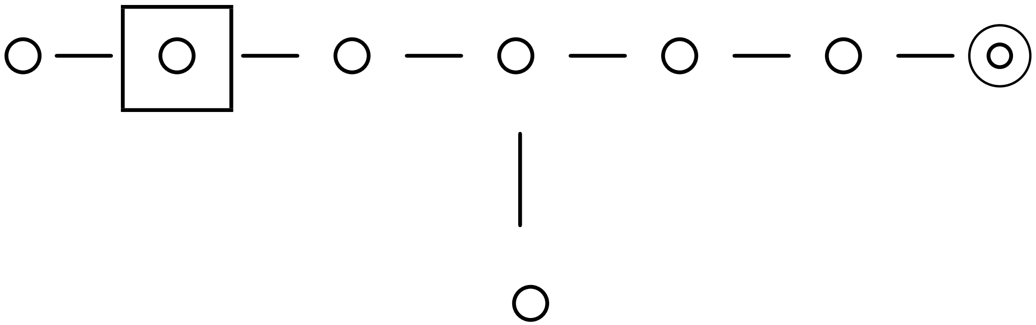
associated the normal triality algebra $A = g_{-1}$ and $Der\ (g_{-1} \oplus\ g_1) \cong g_{-2} \oplus g_0 \oplus g_2$.
From § 6.2 (I) type, we obtain bisymmetric space's series of type (I). It is said to be a type (I) bisymmetric space.

$$F_4/B_4 < E_6/(D_5 \oplus gl(1)) < E_7/(D_6 \oplus A_1) < E_8/D_8$$

dimensin of bisymmetric spaces of type (I);

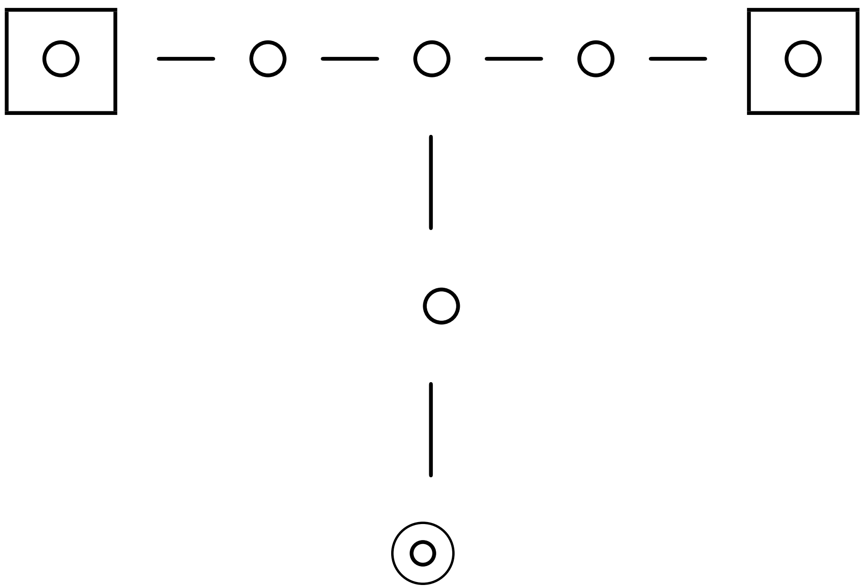
$$16, \quad 32, \quad 64, \quad 128 \qquad respectively.$$

With respect to Extended Dynkin diagrams (symmetric spaces) of E_7, E_6, F_4;



, $\boxed{\circ}$ omitted

$\cong D_6 \oplus A_1$, and $\circledcirc$ is highest root.



, $\boxed{\circ}$ omitted

$\cong D_5 \oplus gl(1)$, and $\circledcirc$ is highest root.

$\circledcirc - \circ - \circ => \circ - \boxed{\circ}$, $\boxed{\circ}$ omitted

$\cong B_4$, and $\circledcirc$ is highest root.

From § 6.2 (II) type, we may define the same concept to type (I) as follows.

$$(II) \qquad (dim\ B_\delta + 16)/dim\ B_\gamma =$$

$$(dim\ B_\gamma + 16)/dim\ B_\beta =$$

$$(dim\ B_\beta + 16)/dim\ B_\alpha = 2,$$

$$and\ B_\alpha < B_\beta < B_\gamma < B_\delta$$

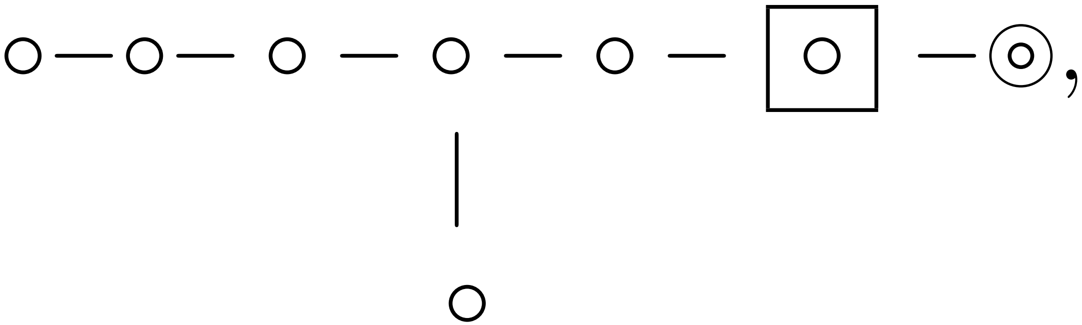
It is said to be a type(II) bisymmetric space.

$$F_4/(C_3 \oplus A_1) < E_6/(A_5 \oplus A_1) < E_7/(D_6 \oplus A_1)$$
$$< E_8/(E_7 \oplus A_1)$$

and the dimension of bisymmetric spaces of type (II) ;

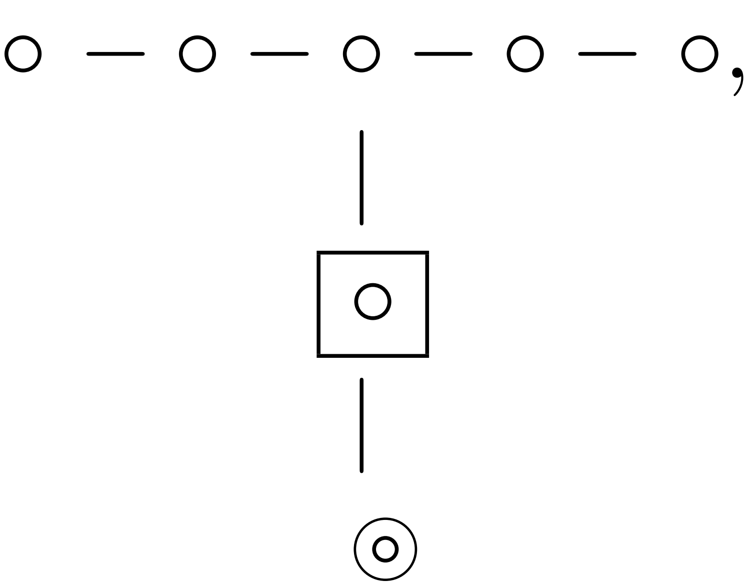
$$28, \quad 40, \quad 64, \quad 112 \qquad respectively.$$

With respect to Extended Dynkin diagrams (symmetric spaces) type (II) of E_7, E_6, F_4;



$\boxed{\circ}$ omitted

$\cong D_6 \oplus A_1$, and $\odot$ is highest root.



$\boxed{\circ}$ omitted

$\cong A_5 \oplus A_1$, and $\circledcirc$ is highest root.
$\circledcirc - \boxed{\circ} - \circ => \circ - \circ$, $\boxed{\circ}$ omitted $\cong C_3 \oplus A_1$.
and $\circledcirc$ is highest root.

Here $A_1, A_5, B_4, C_3, D_5, D_6$ mean classical simple Lie algebras.

Remark For type (II), we consider with $L(A)/g_0$ vector spaces series;

$$F_4/(C_3 \oplus gl(1)) < E_6/(A_5 \oplus gl(1)) < E_7/(D_6 \oplus gl(1))$$

$$< E_8/(E_7 \oplus gl(1))$$

$(dim\ B_\delta + 18)/dim\ B_\gamma =$
$(dim\ B_\gamma + 18)/dim\ B_\beta =$
$(dim\ B_\beta + 18)/dim\ B_\alpha = 2,$
$and\ B_\alpha < B_\beta < B_\gamma < B_\delta.$

This bivector spaces series have dimensions 30, 42, 66, 114 respectively, (may be, it seems that there is a certain algebraic structure, perhaps, to be said a bireductive homogeneous space).

Concluding Remark One of fundamntal our philosophy is to study the construction of

5-graded Lie (super)algebras $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$, satisfying $[g_i, g_j] \subseteq g_{i+j}$without using roots systems and Cartan matrix.

In the end of this paper, it seems that it is useful to refer a Springer publisher book (Math and Statistic series, Lecture notes) with respect to the Proceeding of conferences of Nonassociative algebras and its applications in Coimbra University (2022, Portgal).

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These references are mainly papers for our study fields (as a survey article in these fields, so we have a lot of references).

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Thank you for your attention

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