Proceedings of the Exchange of Mathematical Ideas

2023

Exchange of Mathematical Ideas Conference 2023

Department of Mathematics University of Northern Iowa
Cedar Falls, IA
August 11-13, 2023

The Exchange of Mathematical Ideas (EMI) conference seeks to enhance collaboration and research stimulation among mathematicians across various disciplines. Organized annually by the mathematics departments at Embry-Riddle Aeronautical University Prescott (ERAU) and University of Northern Iowa (UNI), the ninth edition of the conference was held at UNI, Cedar Falls, Iowa. Supported by NSF (award number 2322922), this year's conference spotlighted Algebra and its interconnected domains, featuring expert speakers presenting their research insights.

Organizers
Douglas Mupasiri, UNI
Sirani Perera, ERAU (Daytona)
Michael Prophet, UNI
Hisaya Tsutsui, ERAU

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Christopher Briggs, ERAU

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https://math-dept.info/EMI

10.5281/zenodo.10428467

PLENARY SPEAKERS

Speaker: Daniel B Szyld, Temple University

Title: *Matrices with Perron-Frobenius Properties*

Abstract: Non-nilpotent nonnegative matrices have a positive dominant eigenvalue that corresponds to a nonnegative eigenvector. This property is called the Perron-Frobenius property. General matrices with a Perron-Frobenius property are studied, i.e., matrices which have a positive dominant eigenvalue, with the corresponding eigenvector being positive or non-negative. We concentrate on matrices which are not necessarily non-negative, and whose powers are not necessarily non-negative. Several characterizations of matrices having Perron-Frobenius properties are presented, including some depending on spectral, combinatorial, and geometric characteristics. We also study generalizations of M-matrices, i.e., matrices of the form $sI - B$ with B having a Perron-Frobenius property, and whose spectral radius is no larger than s.

Speaker: Yasuyuki Hirano**,** Hiroshima Institute of Technology

Title: *On Finite Rings and Their Groups of Units*

Abstract: A finite ring is an associative ring consisting of only finitely many elements. Let R denote a finite ring with identity. The group of units in R is denoted by $R^{\times}$. For a set S, we denote the number of elements in S by |S|. In this paper, we analyze the structure of a finite ring R in relation to the number $|R| / |R^{\times}|$. We also study conditions for $R^{\times}$ to be a simple group.

Speaker: John A. Beachy, Northern Illinois University

Title: *Universal localization at semiprime Goldie ideals*

Abstract: My talk will mostly be expository, in an effort to call attention to the method of noncommutative localization introduced by P. M. Cohn (in 1973). Given a prime ideal P of a Noetherian ring R, he constructed the ring universal with respect to inverting the set of matrices inverted by the canonical mapping from
R to the classical ring of quotients of R/P. This ring always exists, but the definition via a universal property seems to only provide information about the "top" of the localized ring, while it remains difficult to even determine the kernel of the localizing homomorphism. I would note that Cohn's construction reduces to the Ore localization when the prime ideal is localizable, and it turns out to be closely related to the localization defined earlier by Alfred Goldie (in 1967). On the other hand, it seems to be on the opposite end of some spectrum involving the torsion theoretic localization. There are many "known" unknowns, along with what is surely a large number of "unknown" unknowns. I still have hope that Cohn's method can provide a language that will help in extending commutative results involving localization to the noncommutative case.

Speaker: Noriaki Kamiya, Affiliation: University of Aizu

Title: *On certain algebraic structures associated with Lie (super)algebras*

Abstract: This talk is to deal with a certain survey and to several examples of triple systems, furthermore to describe a history of Jordan river in nonassociative algebras from the author's viewpoint. Also, we will speak a correspondence with Lie (or Jordan) structures and symmetric spaces with complex structure.

Speaker: Steven H. Weintraub, Lehigh University

Title: *Reverse Orthogonal Polynomials*

Abstract: Let $\mathscr{P}$ be the vector space of all polynomials equipped with an inner product. We may apply the Gram-Schmidt procedure to the ordered basis $\{1,\ x,\ x^2,...\}$. of $\mathscr{P}$, with suitable normalization, to obtain orthogonal polynomials $\{F_0(x),\ F_1(x),\ F_2(x),...\}$. This is a classical construction. Instead, for any fixed *n*, we begin with the vector space $\mathscr{P}_n$ of polynomials of degree at most *n*, with the restriction of the same inner product, and apply the Gram-Schmidt procedure to the ordered basis $\{x^n, x^{n-1},\ ...,1\}$ of $\mathscr{P}_n$, with suitable normalization, to obtain orthogonal polynomials $\{\overset{\leftarrow n}{F}_n(x),\ \overset{\leftarrow n}{F}_{n-1}(x),\ ...,\ \overset{\leftarrow n}{F}_0(x)\}$. Since we are applying the Gram-Schmidt procedure in reverse order, we call these reverse orthogonal polynomials. We discuss the reverse orthogonal polynomials, with particular emphasis on the reverse Legendre polynomials and the reverse Chebyshev polynomials of the first and second kinds.

Participants

Alvarez, Angelynn, ERAU - Prescott
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Briggs, Christopher, ERAU - Prescott
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Watanabe, Tatsunari, ERAU - Prescott
Weintraub, Steven, Lehigh University
Wood, William, University of Northern Iowa
Xie, Weiguo, University of Minnesota - Duluth
Zufelt, Mitchell, University of Chicago

Schedule

Friday, August 11, 2023

Starting time	Title of the talk and presenter
9:15 am	Welcome: Jose Herrera, Provost, University of Northern Iowa (UNI) Opening Remarks: Doug Mupasiri, Head, UNI Department of Mathematics
9:30am – 10:10am	Title: *Harris Graphs – an introduction* Speaker: Douglas Shaw (UNI)
10:25am – 11:05am	Title: *The Geometry of the Hyperelliptic Torelli Group* Speaker: Tatsunari Watanabe (ERAU-Prescott)
	Conference Lunch
1:15pm – 2:15pm	Plenary Talk 1 Title: *Matrices with Perron-Frobenius Properties* Speaker: Daniel Szyld (Temple University)
2:30pm – 3:10pm	Title: Dehn Invariant Zero Tetrahedra Speaker: Anas Chentouf (MIT)
	Tea/Coffee Break
3:50 pm - 4:30pm	Title: *A Low-Cost Algorithm to Determine Orbital Trajectories within Cislunar Region* Speaker: Sirani Perera (ERAU – Daytona Beach)
4:45pm – 5:15pm	Title: Uniform exponential growth of Lie algebras and their associated universal enveloping algebras Speaker: Christopher Briggs (ERAU, Prescott) – via Zoom

Saturday, August 12, 2023

9:20am –10:00am	Title: *Geometric Analysis Under Ricci Curvature Bounds* Speaker: Shoo Seto (California State University, Fullerton)
10:15am – 10:55am	Title: *Rotation Operations on the Errera Map and its Variations – Part 1* Speaker: Weiguo Xie (University of Minnesota Duluth)
11:10am – 12:10pm	Plenary Talk 2 Title: *On Finite Rings and Their Groups of Units* Speaker: Yasuyuki Hirano (Hiroshima Institute of Technology, Japan)
	Conference Lunch
1:15pm – 1:55pm	Title: *DSR-graphs and group algebras* Speaker: Tsunekazu Nishinaka (University of Hyogo, Japan)

2:10pm – 2:50pm	**Title:** ***Introduction to Fully Prime Rings*** **Speaker: Hisaya Tsutsui (ERAU-Prescott)**
	Tea/Coffee Break
3:20pm – 4:20pm	**Plenary Talk 3** **Title:** ***Universal localization at semiprime Goldie ideals*** **Speaker: John Beachy (Northern Illinois University)**
4:35pm – 5:35pm	**Poster Session:** • **Ehiareshan Obeason (Illinoi State University)** • **Kodjo Houssou (University of Minnesota Twin City)** • **Cesiley Barnes (Illinoi State University)**
7:00pm -	**Conference Dinner** **Venue: George's Local, 108 E 4th St, Cedar Falls, IA 50613**

Sunday, August 13, 2023

9:30am –10:10am	**Title:** ***Rotation Operations on the Errera Map and its Variations – Part 2*** **Speaker: Andrew Bowling (University of Minnesota Duluth)**
11:20am – 12:20pm	**Plenary Talk 4** **Title:** ***On certain algebraic structures associated with Lie (super)algebras*** **Speaker: Noriaki Kamiya (University of Aizu, Japan)**
	Conference Lunch
1:15pm – 1:55pm	**Title: Semi-Implicit Time Integration for Partial Differential Equations and The Method of Regularized** **Stokeslets** **Speaker: Benjamin Stager (Tulane University) – via Zoom**
2:10pm – 3:10pm	**Plenary Talk 5** **Title:** ***Reverse Orthogonal Polynomials*** **Speaker: Steven H. Weintraub (Lehigh University)**
	Closing Remark by organizers.

FOREWORD

The International Conference for the Exchange of Mathematical Ideas was started by three founding organizers: Douglas Mupasiri of University of Northern Iowa, Keith Mellinger of University of Mary Washington, and Hisaya Tsutsui of Embry-Riddle Aeronautical University.The first conference took place at Embry-Riddle Aeronautical University's Prescott campus on May 26, 2012. It had an international audience of 21 participants representing diverse mathematical specialties ranging from noncommutative ring theory to computability theory, cryptography to topology, algebraic number theory to operator theory.

The ethos of the conference is grounded on recognition of the surprising connections that arise between distant fields. That by getting together to describe their research to an audience of non-specialists, researchers often gain new perspective on their own work and find inspiration in the work of others. The EMI is intended to provide a venue for mathematicians to interact in this way. Indeed, collaborations across disciplines sparked at the Exchange have resulted in research productivity, including peer-review journal publications.

Most of all, even though mathematics can be done alone, and often is the product of individual effort, it gains meaning only when shared. We gather to pay homage to this communal aspect of mathematics. We dedicate these proceedings to those who have been with us in the past and those who will join us in the future.

Participants of the 2023 meeting were invited to submit papers to the proceedings. The seven selected submissions are published here.

Contents

THE GEOMETRY OF THE HYPERELLIPTIC TORELLI GROUP

TATSUNARI WATANABE

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Abstract. In this survey paper, we will introduce the hyperelliptic mapping class group of an oriented topological surface and its certain infinite-index subgroup called the hyperelliptic Torelli group. These groups are subgroups of the mapping class group, which has been extensively studied in many areas of mathematics. The hyperelliptic involution yields a symmetry on the surface and determines the hyperelliptic mapping class group. In this paper, we introduce the symmetry and introduce some open problems in the study of the hyperelliptic mapping class group and the hyperelliptic Torelli group.

1. Introduction

It is often said that a coffee mug is topologically the same as the torus. This is because a mug can be deformed to the torus as in Figure 1. In Figure 1, the simple closed curve γ is transformed to the curve α in the torus, while in Figure 2, the curve γ is transformed to a different simple closed curve β. So, intuitively, we see that these two deformations are distinct. In topology, we are interested in properties that are invariant under all continuous deformations. Furthermore, a deep and difficult problem of our interest is the study of the relations between the geometric properties of a topological surface and the group of the deformations of the surface.

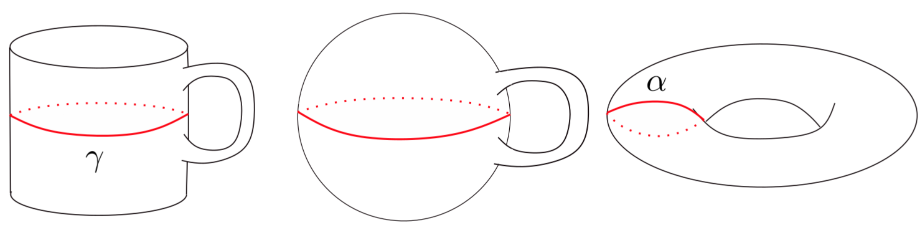

Figure 1. A deformation of a mug to the torus

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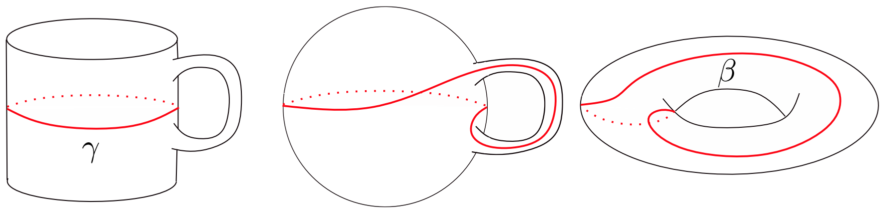

FIGURE 2. Another deformation of a mug to the torus

To be more precise, by a deformation, we mean a diffeomorphism, which is a smooth invertible map between topological surfaces which admits a smooth inverse. The mapping class group of an oriented topological surface S is the group of isotopy classes of orientation-preserving diffeomorphisms of S. It has been studied extensively in many areas of mathematics such as topology, geometry, algebraic geometry, number theory, and etc. A hyperelliptic involution σ of S is an orientation-preserving diffeomorphism of order 2. The hyperelliptic mapping class group of S is the centralizer of the isotopy class of σ in the mapping class group, and the symmetry produced by σ should be captured in its group structure.

The mapping class group acts on the fundamental group of S, which yields the symplectic representation. The kernel of the representation is called the Torelli group. The intersection of the Torelli group and the hyperelliptic mapping class group is called the hyperelliptic Torelli group. In this notes, we will introduce the generators of the hyperelliptic mapping class group and the hyperelliptic Torelli group, and some of open problems in the study of these groups.

2. Topology of curves

A complex curve C is a smooth projective irreducible variety of dimension one defined over the complex numbers $\mathbb{C}$. The curve C of genus $g \geq 0$ is diffeomorphic to a compact oriented topological surface, denoted by S_g, with g holes. Fix a point p in C. The fundamental group $\pi_1(C,p)$ of S with base point p is the group of homotopy classes of loops in C based at p. A different choice of base point q yields a natural isomorphism

$$\pi_1(C,p) \cong \pi_1(C,q),$$

which is unique up to a conjugation action by an element of $\pi_1(C,p)$. Therefore, we omit the base point from the notation. Let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be the standard generating set for $\pi_1(C)$. It has a minimal presentation given by

$$\pi_1(C) = \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \,\middle|\, \prod_{j=1}^{g} [\alpha_j, \beta_j] \right\rangle.$$

The natural map $\pi_1(C) \to H_1(C,\mathbb{Z})$ induces an isomorphism from the abelianization of $\pi_1(C)$ to the homology group $H_1(C,\mathbb{Z})$. Denote the images of α_j and β_j in $H_1(C,\mathbb{Z})$ by a_j and b_j for $j = 1, \ldots, g$. The abelianization $H_1(C,\mathbb{Z})$ is a free abelian group of rank $2g$.

2.1. **Symplectic group.** The group of symplectic matrices of size $2g \times 2g$ over $\mathbb{Z}$ is denoted by $\mathrm{Sp}(2g;\mathbb{Z})$. It is defined as the group of invertible $2g$-by-$2g$ matrices M with entries in $\mathbb{Z}$ satisfying

$$M^T J M = J,$$

where $J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$ and I_g is the g-by-g identity matrix.
The group $H := H_1(C,\mathbb{Z})$ is equipped with the algebraic intersection paring $\langle\ ,\ \rangle$. The pairing $\langle\ ,\ \rangle$ is a non-degenerate bilinear alternating form, and H is a symplectic space of rank $2g$. Fix a symplectic basis $a_1,\dots,a_g,b_1,\dots,b_g$ for H. Then there is an isomorphism of the automorphism of H preserving $\langle\ ,\ \rangle$ with $\mathrm{Sp}(2g;\mathbb{Z})$:

$$\mathrm{Aut}(H,\langle\ ,\ \rangle) \cong \mathrm{Sp}(2g;\mathbb{Z}).$$

3. Mapping Class Groups

Let S_g be a compact oriented surface of genus g. Denote the punctured surface obtained from S_g by removing n distinct points by $S_{g,n}$. The mapping class group of $S_{g,n}$, denoted by $\Gamma_{g,n}$, is defined as the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ fixing the punctures pointwise. The n punctures are often viewed as marked points on S_g as well. The group $\Gamma_{g,n}$ is independent of a choice of the n points removed by the classification of surfaces. When $n = 0$, we simply denote $\Gamma_{g,0}$ by Γ_g. In this notes, we always assume that $2g - 2 + n > 0$.

3.1. **Dehn twists.** When $n = 0$, the group Γ_g is finitely generated by the isotopy classes of a certain type of diffeomorphisms called Dehn twists. A Dehn twist T_d about a simple closed curve d in S_g is a left-twist map about d, fixing the boundary of a tubular neighborhood N of d. More precisely, let A be the cylinder oriented outward given by $S^1 \times [0,1]$ with coordinates θ and t, respectively. Let $T : A \to A$ be the twisting map sending $(\theta,t) \mapsto (\theta + 2\pi t, t)$. Note that T is an orientation-preserving diffeomorphism fixing the boundary of A pointwise. Choose an orientation-preserving diffeomorphism $\psi : A \to N$. Define a map $T_d : S_g \to S_g$ by sending

$$x \mapsto \begin{cases} \psi \circ T \circ \psi^{-1}(x) & \text{if } x \text{ is in } N, \\ x & \text{otherwise.} \end{cases}$$

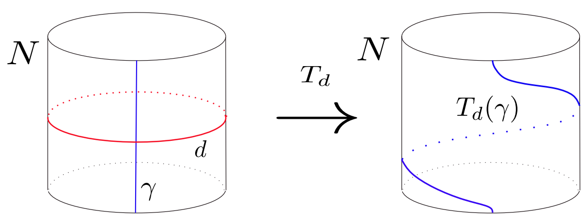

Figure 3. A Dehn twist

A simple closed curve d in S_g is said to be separating if the surface obtained by cutting S_g along d is disconnected. Otherwise, it is said to be nonseparating. When $g = 1$, Γ_g is generated by the Dehn twists about α_1 and β_1 in the torus. For $g \geq 2$, the mapping class group Γ_g is finitely generated by the isotopy classes of Dehn twists about $2g+1$ nonseparating simple closed curves in S_g ([3, Thm. 4.14]). Furthermore, it is also finitely presented [3, Thm. 5.3].

3.2. **Symplectic representation of $\Gamma_{g,n}$.** Fix p in S_g. The mapping class group $\Gamma_{g,n}$ acts on $\pi_1(S_g)$, and furthermore this action induces an action on H, preserving the intersection pairing $\langle\ ,\ \rangle$. Hence we obtain a homomorphsim

$$\rho_{g,n} : \Gamma_{g,n} \to \mathrm{Sp}(2g; \mathbb{Z}).$$

The homomorphism $\rho_{g,n}$ is surjective for $g \geq 1$ [3, Thm. 6.4].

3.3. **The Birman exact sequences.** There is a natural injection $\mathcal{P}\mathrm{ush} : \pi_1(S_g) \hookrightarrow \Gamma_{g,1}$ called the push map:

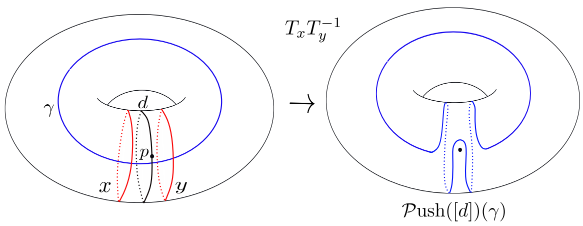

FIGURE 4. The point-pushing map $\mathcal{P}\mathrm{ush}$

Combining with the forgetful map $\mathcal{F}\mathrm{orget} : \Gamma_{g,1} \to \Gamma_g$, we obtain the Birman exact sequence:

$$1 \to \pi_1(S_g) \to \Gamma_{g,1} \to \Gamma_g \to 1.$$

The Birman exact sequence extends to the puncture case: the sequence

$$1 \to \pi_1(S_{g,n}) \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1$$

is exact.

Theorem 3.1. [3, Cor. 5.11] *If $g \geq 2$, the Birman exact sequence does not split.*

It is well known that the punctured Birman exact sequence does not split either. For example, it follows from [12, Theorem 1].

3.4. **Torelli group.** The Torelli group is defined to be the kernel of the symplectic representation $\rho_{g,n}$: $T_{g,n} = \ker \rho_{g,n}$:

$$1 \to T_{g,n} \to \Gamma_{g,n} \overset{\rho_{g,n}}{\to} \mathrm{Sp}(2g; \mathbb{Z}) \to 1.$$

It is an infinite-index subgroup of $\Gamma_{g,n}$. Therefore, there is no reason to believe that it carries the basic properties of $\Gamma_{g,n}$.

4. Hyperelliptic Mapping Class Groups

A hyperelliptic involution $\sigma : S_g \to S_g$ is an orientation preserving diffeomorphism of order 2 of S_g fixing exactly $2g+2$ points.

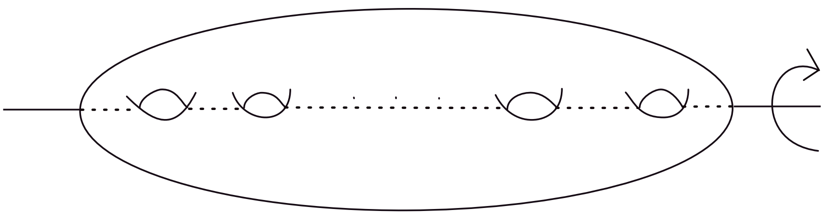

Figure 5. A hyperelliptic involution of S_g, rotation by π

Fix a hyperelliptic involution σ of S_g.

Definition 4.1. The hyperelliptic mapping class group Δ_g of S_g is defined to be

$$\Delta_g := \text{the centralizer of the isotopy class of } \sigma \text{ in } \Gamma_g$$

The hyperelliptic mapping class group $\Delta_{g,n}$ of type (g,n) is defined to be

$$\Delta_{g,n} := \Delta_g \times_{\Gamma_g} \Gamma_{g,n}$$

4.1. **Generators.** A simple closed curve γ is said to be symmetric if $\sigma(\gamma) = \gamma$. The hyperelliptic mapping class group Δ_g can be generated by the Dehn twists about $2g+1$ symmetric nonseparating simple closed curves:

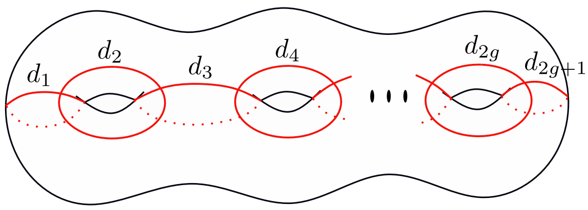

Figure 6. Nonseparating symmetric curves generating generating Δ_g

4.2. **Hyperelliptic Torelli group.**

Definition 4.2. The hyperelliptic Torelli group $T\Delta_g$ is defined to be the intersection of Δ_g with T_g:

$$T\Delta_g := \Delta_g \cap T_g.$$

Theorem 4.3 (Brendle-Margalit-Putman)**.** *If $g \geq 2$, then $T\Delta_g$ is generated by Dehn twists about symmetric separating curves.*

Remark 4.4. When $g = 2$, any two simple separating curves intersect at least 4 times (see Figure 7). On the other hand, when $g \geq 3$, there are disjoint symmetric separating curves as in Figure 8, which produce commuting Dehn twists in $T\Delta_g$.

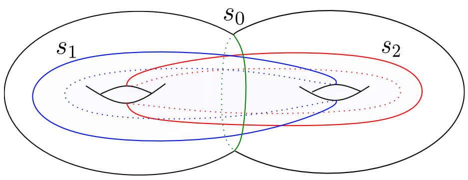

FIGURE 7. Symmetric separating curves in S_2

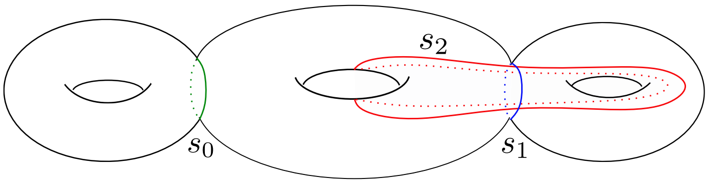

FIGURE 8. Symmetric separating curves in S_3

4.3. **Open problems.** Wile the Torelli group T_g is known to be finitely generated for $g \geq 3$, it is still an open question for the hyperelliptic Torelli group $T\Delta_g$:

Is $T\Delta_g$ finitely generated for $g \geq 3$?

When $g = 2$, Mess proved in [7] that $T\Delta_2 = T_2$ is an infinitely generated free group.

Similarly, the finite presentability of $T\Delta_g$ is not known:

Is $T\Delta_g$ finitely presentable for $g \geq 4$?

From the cohomological analysis of $T\Delta_g$ done by Brendle, Childers, and Margalit in [2], it is known that $T\Delta_3$ is not finitely presentable.

Another important question is the abelianization of $T\Delta_g$:

Determine $H_1(T\Delta_g)$.

Johnson computed in [6] the abelianization of T_g, and the result has played an important role in many different areas of algebraic geometry such as the study of rational points of universal curves (see [4], [11]). The analogous result for the hyperelliptic Torelli group should be an important tool for the study of the universal hyperelliptic curve, which is the restriction of the universal curve to the hyperelliptic locus in the moduli of curves. For example, the author studies the sections of the universal hyperelliptic curves in [10] using the relative completion of the hyperelliptic mapping class groups. The structure of the completion has a deep connection with the abelianization of the hyperelliptic Torelli group, and it can be used to study the geometric properties of the universal curves.

References

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GEOMETRIC ANALYSIS OF CURVATURE ON RIEMANNIAN MANIFOLD

SHOO SETO

Abstract. We give a brief and informal survey of the notion of curvatures on a Riemannian manifold. As a particular example to focus on, we will introduce the eigenvalue problem of the Laplacian and how the Ricci curvature will be involved.

1. Brief Introduction to Curvature

The study of "curvature" has been a central topic in differential geometry since its inception. Intuitively, we see that a "curved" surface is different from a "non-curved" planar surface. To precisely define what it means to be a "curvature" is delicate. Do we associate some scalar quantity as curvature of a surface? Here we give a brief and informal discussion on curvature.

In the one-dimensional case, we define the curvature of a unit-speed regular curve γ is the scalar quantity $\kappa = |\gamma''(t)|$. Then if $\kappa = 0$, we have $\gamma'(t)$ is constant, hence $\gamma(t)$ is simply a straight line, which agrees with our intuitive notion of "curvedness". The ambiguities of curvedness begin in the two-dimensional case. For surfaces, we can consider some guiding examples, such as a "flat" sheet of paper, a parabolic cylinder, and a sphere.

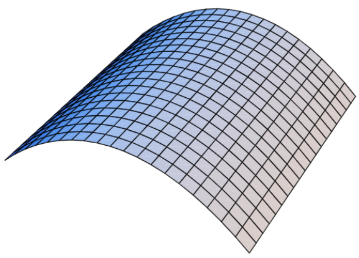

Figure 1. A Parabolic Cylinder

To build up from the one-dimensional case, we consider curves of intersection with a plane normal to the surface at a point. Such curves are called normal sections and at the point where the normal plane was determined, we assign scalar value called normal curvature. The scalar value here is not exactly the curvature from the one-dimensional case as we allow a sign. On a plane, it is easily seen that all values of the normal curvature is zero, hence matches with our intuition that a "flat" plane should have "zero curvature". For the parabolic cylinder, we notice two special normal sections. The one tangent in the direction of the parabola maximizes the normal curvature and the direction perpendicular minimizes since the normal section is a straight line hence zero curvature. These are called the principal curvatures, denoted μ_1 and μ_2, and from these we define the mean curvature $H = \frac{\mu_1+\mu_2}{2}$ and the Gauss curvature $K = \mu_1\mu_2$. As a first observation, it is not difficult to see that $K = 0$ for the flat plane and the parabolic cylinder. Applying the same construction to the sphere, we see that $K > 0$. Here we see glimpses of how the surface of the sphere is fundamentally "different" from that of the plane and of the parabolic cylinder. In fact, the Gaussian curvature remains invariant under isometry transformations of the surface.

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We imagine taking a sheet of paper and slightly folding it to construct a parabolic cylinder. This transformation leaves the distance *along the surface* of two points the same. From this we see that it is impossible to make a perfect world map onto a flat sheet of paper.

The principal curvatures do not generalize well to general n-dimensional spaces since we need a notion of a "normal" direction. For embedded hypersurfaces $M^n \subset \mathbb{R}^{n+1}$, this is possible but in general we are not. Nevertheless, the Gauss curvature does generalize to n-dimensional Riemannian manifolds. On Riemannian manifolds, at each point we have a vector space of possible directions, the tangent space. This gives us (in a rough sense) a way to take directional derivatives of various tensor quantities, including the (tangent) vector fields themselves. With this, we can define a notion of curvature in the following way. Let $\mathfrak{X}(M)$ be the space of vector fields on M.

Definition 1.1. Define the $(3,1)$-Riemann Curvature Tensor $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$\begin{aligned} R(X,Y,Z) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z \end{aligned}$$

where $[X,Y] = XY - YX$ is the Lie bracket.

Informally, the curvature tensor measures the non-commutativity of the directional derivative (more precisely, the covariant derivative).

On Riemannian manifolds, we have an inner product defined on the tangent space at each point p, and they vary smoothly with respect to p. Using the metric, we have a rank $(4,0)$ Riemann curvature tensor:

Definition 1.2. Let $X, Y, Z, W \in T_pM$ and $R(X,Y)Z \in T_pM$ be the $(3,1)$-Riemann Curvature tensor. Then

$$R(X,Y,Z,W) = g(R(X,Y)Z, W).$$

In coordinates we have $R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl}$. Taking certain sums of the curvature tensor, we define

Definition 1.3. For each 2-plane $P \subset T_pM$, the sectional curvature $K(P)$ is defined by

$$K(P) = R(X_1, X_2, X_2, X_1)$$

where $\{X_1, X_2\}$ is an orthonormal basis for P.

Using the symmetries of the curvature tensor, the sectional curvature fully determines the Riemann curvature tensor.

If M is a 2-dimensional Riemannian manifold, then the sectional curvature is equal to the Gauss curvature.

We can also take the trace of the curvature tensor.

Definition 1.4. The Ricci curvature tensor is the "trace of the Riemann tensor" or more precisely for $X, Y, Z \in T_pM$,

$$\mathrm{Ric}(Y,Z) := \mathrm{Tr}(X \mapsto R(X,Y)Z).$$

If $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_pM, then

$$\mathrm{Ric}(Y,Z) := \sum_{i=1}^{n} R(e_i, Y, Z, e_i).$$

It is this Ricci curvature which we will focus on for the remainder of this article.

2. Eigenvalues of the Laplacian

In this section, we introduce a differential operator of fundamental importance in geometric analysis, the Laplace-Beltrami operator, or the Laplacian, and its corresponding eigenvalue problem.

Let (M^n, g) be an n-dimensional Riemannian manifold, possibly with boundary. The Laplacian Δ is a differential operator defined on twice-differentiable functions on M given by the divergence of the gradient of a function u, i.e.,

$$\Delta u := \operatorname{div}(\nabla u).$$

Equivalently, the Laplacian is given as the trace of the Hessian of u. The Laplacian arises in the study of differential equations describing many different physical phenomena and its importance in the field of mathematics and physics cannot be understated.

The behavior of the Laplacian is highly dependent on the underlying geometry and can be seen in a quantitative sense by the Bochner formula. For $u \in C^3(M)$, we have

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\operatorname{Hess} u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \operatorname{Ric}(\nabla u, \nabla u).$$

Now we consider the L^2-space of square integrable real-valued smooth functions $C^\infty(M)$. The smoothness assumption is stronger than what is needed however we will assume it for convenience. We can equip the space with the L^2-inner product

$$\langle u, v\rangle_{L^2} := \int_M uv dV.$$

By applying the divergence theorem, we see that the Laplacian is a self-adjoint operator on the L^2-space. If the underlying space M is compact, then by the spectral theory of compact self-adjoint operators, the spectrum of Δ consists of eigenvalues (starting possibly at either λ_0 or λ_1 to be elaborated later)

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \lambda_k \to \infty$$

of finite multiplicity. As a PDE, we write $\Delta u = -\lambda u$, with $u \in H$, some Hilbert space of functions. In this article, we will focus on the case that M is a closed (compact and no boundary) manifold. In this case, the smallest eigenvalue is 0 since $u \equiv c \neq 0$, c is some constant, will serve as a solution to the eigenvalue problem. For the first non-trivial eigenvalue, we have the following extremal characterization.

$$\lambda_1 = \inf\left\{ \frac{\int_M |\nabla u|^2 dV}{\int_M u^2 dV} \;\middle|\; \int u dV = 0, u \not\equiv 0 \right\}.$$

From this extremal characterization, we have the Poincaré inequality: For $u \in C^1(M)$ such that $\int_M u dV = 0$, we have

$$\lambda_1 \int_M u^2 dV \le \int_M |\nabla u|^2 dV.$$

Knowing the value of λ_1 will give us the optimal Poincaré constant however the eigenvalues are computable only in special cases often involving some symmetry. A non-zero lower bound will suffice to obtain the inequality and is what we will focus on here.

There are two classical results for a lower bound of the first nontrivial eigenvalue.

Theorem 2.1 (Lichnerowicz [4])**.** Let (M, g) be a complete n-dimensional Riemannian manifold with $\operatorname{Ric} \ge (n-1)K > 0$. Then

$$\lambda_1 \ge nK.$$

Remark 2.1. Obata [5] establishes that equality holds if and only if M is isometric to a round n-sphere of radius $\frac{1}{\sqrt{K}}$ is equal to nK

When $K = 0$, the Lichnerowicz estimate does not give us new information. In such a case, we have the following estimate of Zhong-Yang.

Theorem 2.2 (Zhong-Yang [7]). Let (M, g) be a closed Riemannian manifold with $\mathrm{Ric} \geq 0$. Then

$$\lambda_1 \geq \frac{\pi^2}{D^2},$$

where $D = \mathrm{diam}(M)$.

Remark 2.2. Hang and Wang [2] established that equality holds if and only if M is isometric to S^1 with radius $\frac{D}{\pi}$.

The two classical estimates mentioned above has been unified as a comparison result between the first nonzero eigenvalue on M and the first nonzero eigenvalue of an ODE.

Theorem 2.3 (Kröger [3], Bakry-Qian [1]). Let (M, g) be a compact n-dimensional Riemannian manifold (possibly with a smooth convex boundary) with diameter D and $\mathrm{Ric} \geq (n-1)K$ for $K \in \mathbb{R}$. We assume Neumann boundary conditions if $\partial M \neq \emptyset$. Then

$$\lambda_1 \geq \bar{\lambda}_1(n, K, D)$$

where $\bar{\lambda}_1(n, K, D)$ is the first nonzero Neumann eigenvalue of the one-dimensional eigenvalue problem

$$\varphi'' - (n-1)T_K\varphi' = -\bar{\lambda}\varphi$$

on the interval $[-D/2, D/2]$. Here the function T_K, $K \in \mathbb{R}$ is defined to be

$$T_K(x) := \begin{cases} \sqrt{K}\tan(\sqrt{K}x), & K > 0 \\ 0 & K = 0 \\ -\sqrt{-K}\tanh(\sqrt{-K}x) & K < 0. \end{cases}$$

The above is indeed a uniformization of Lichnerowicz and Zhong-Yang estimates since when $K = 0$, the explicit solution is given by $\varphi(x) = \cos(\frac{\pi}{D}x)$. When $K > 0$, we can consider the model when $D = \frac{\pi}{\sqrt{K}}$ so that the eigenfunction is given by $\varphi(x) = \sin(\sqrt{K}x)$ which has the eigenvalue $\bar{\lambda} = nK$.

3. Integral Ricci Curvature

We now generalize the pointwise lower bound to an integral condition. Let $\rho(x)$ be the smallest eigenvalue for the Ricci tensor and let $\rho_K := \max\{-\rho(x) + (n-1)K, 0\}$. Define the quantity

$$\bar{k}(p, K) := \left(\fint_M \rho_K^p dV \right)^{\frac{1}{p}}$$

which measures the amount of Ricci curvature lying below $(n-1)K$ in an L^p average sense. Note that $\bar{k}(p, K) = 0$ iff $\mathrm{Ric} \geq (n-1)K$.

Several results under the pointwise Ricci lower bound have been generalized to the integral Ricci curvature condition case. In particular, we have a generalization of the Zhong-Yang estimate:

Theorem 3.1 (Ramos Olivé-Seto-Wei-Zhang [6])**.** Let (M^n, g) be a closed Riemannian manifold with diameter $\leq D$ and λ_1 be the first nonzero eigenvalue. For any $\alpha \in (0,1)$, $p > \frac{n}{2}$, $n \geq 2$, there exist s$\varepsilon(n, p, \alpha, D) > 0$ such that if $\bar{k}(p, 0) \leq \varepsilon$, then

$$\lambda_1 \geq \alpha \frac{\pi^2}{D^2}.$$

With a more delicate analysis, we can generalize the Kröger estimate for the integral Ricci curvature case.

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To Color the Errera Map and its Variations Using Four Colors

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Abstract

A Kempe chain in a colored graph is a maximal connected component containing at most two colors. Kempe chains have played an important role historically in the study of the Four Color Problem. Some methods of systematically applying Kempe chain color exchanges have been studied by Alfred Errera and Weiguo Xie. A map constructed by Errera represents an important counterexample to some implementations of these methods. Using the ideas of Irving Kittell, we determine all colorings of the Errera map which form such a counterexample and describe how to color them individually. We then extend our results from the Errera map to a family of graphs containing the Errera map in a specific way. Being able to color this family of graphs appears to address many cases which prove difficult for the previous systematic color exchange methods.

Key Words: Four Color Problem, cubic map, Kempe chains, Errera map, coloring algorithms

AMS Subject Classification: 05C15.

1 Introduction

A *plane graph G* is a graph drawn in the plane (or on the sphere) in which no two edges cross. In this paper, we explore one of the most famous theorems in all of graph theory relating to plane graphs: the Four Color Theorem. One statement of the Four Color Theorem is given below.

Theorem 1.1. *The regions of a plane graph G can be colored with at most 4 colors such that no two adjacent regions (that is, regions sharing a boundary line) have the same color.*

Such a coloring is known as a *(proper) coloring* of the regions of G. The Four Color Theorem has a rich and fascinating history. In 1879, Alfred Bray Kempe famously (or

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perhaps infamously) attempted to prove the Four Color Theorem using what are now called "Kempe chains." Given a partially colored plane graph G and two colors A, B, we can define an *AB Kempe chain* as a maximal set of connected regions of G containing only colors A and B. A *color exchange* on an AB Kempe chain is the result of permuting the colors A and B on the Kempe chain. Kempe believed that a simple sequence of color exchanges on these connected components would always be able to produce a four coloring of a plane graph from a partial coloring of all regions excepting one region having at most 5 neighbors (see [6, 10]). This approach was ultimately disproven by Heawood in [4]. While there are computer-assisted proofs of the Four Color Theorem, notably those in [1] and later in [9], these are "machine-checkable proofs" and have not been checked by human readers.

While Kempe's own attempt was flawed, the idea of Kempe color exchanges on Kempe chains is much easier to grasp than the computational approaches in [1] and [9], and it has proven very effective in practice. It has been demonstrated in [3, 5, 8] that randomly applying color exchanges to Kempe chains has a very high success rate in being able to color a plane graph in relatively few exchanges (see Section 6 for a more precise description of how these exchanges are used). The question, however, is still open if this method will ever get "stuck," or if using sequences of Kempe chain exchanges will always resolve any issues in coloring. Notably, proof of such a fact would be a proof of the Four Color Theorem, so we anticipate such a proof to be quite challenging.

Instead of applying random color exchanges, we can explore systematic applications of specific color exchanges. This was studied by Alfred Errera in 1921 [2], and more recently by Weiguo Xie in [11, 12, 13]. In both cases the authors consider coloring the regions of *cubic maps*, where a cubic map is a 2-connected 3-regular plane graph. It is known that showing that the regions of all cubic maps can be properly four-colored is equivalent to proving the Four Color Theorem.

In [2], Errera produced a map with a rather unusual property: given a partial coloring of the map so that all but one region (in this case, all but the exterior region) are colored, and given a particular method of routinely applying Kempe color exchanges, the graph returns to its original coloring after 20 such exchanges. Therefore, this coloring fails to reach a state where the exterior region can be colored using this method. As Errera was hoping to prove the Four Color Theorem using this technique, the existence of this counterexample was troublesome. Errera's map is shown in Figure 1, as presented in [7]. For future reference, we will denote this partial coloring of the Errera map c_0.

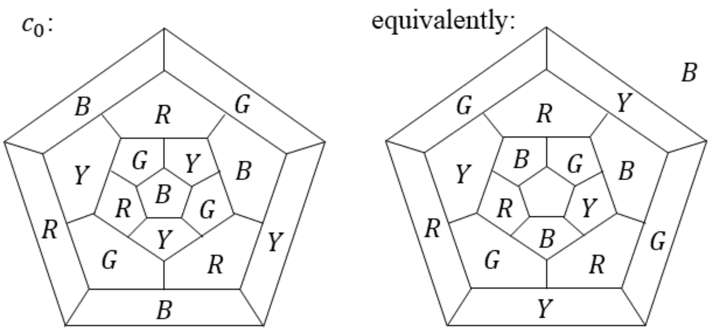

Figure 1: A coloring of the Errera map, providing a counterexample to Errera's original coloring method. In the first graph the uncolored region is the exterior region, while in the second the uncolored region is an interior region. There is no significant difference between these two representations; we will follow [7] and use the first.

In this paper, we will thoroughly examine Errera's counterexample, determine how to resolve the partial coloring into a proper four-coloring of the entire map, and see how this method can be expanded to a larger class of graphs.

2 Definitions

In 1935, Errera's work was expanded by Irving Kittell, who defined eight different ways to perform Kempe color exchanges on the regions of a map (see [7]). The "random color exchanges" mentioned previously are typically applications of Kittell's operations. Of those operations, several will be of interest to us. We will assume in the following definitions that the exterior region is the uncolored region, that the exterior region has five neighboring regions, and that these neighboring regions contain all four colors. (Later, we will modify these assumptions to account for graphs with more than one uncolored region, but they are sufficient for the time being.) This means that one color is used twice amongst the neighboring regions. For brevity, we will call the colored regions adjacent to the uncolored region *boundary regions.* The following notation and definitions are taken from [7], with some rephrasing and adaptation. We will refer to the coloring in Figure 1 for examples.

- The boundary region situated between two boundary regions of the same color is called the *vertex.* In Figure 1, this is the boundary region labeled R.

- A Kempe chain containing both the vertex and and the boundary region positioned two spaces clockwise from the vertex is called the *left-hand circuit.* We may also refer to this circuit by its colors; in Figure 1, we would call the left-hand circuit the *RG circuit.* If such a Kempe chain does not exist, we will refer to the Kempe chain using these colors and starting at the vertex as a *broken left-hand circuit.*

- Similarly, a Kempe chain containing both the vertex and the boundary region positioned two spaces counterclockwise from the vertex is called the *right-hand circuit.* In Figure 1, we would call this the *RY circuit.* We analogously define *broken right-hand circuit.*

- A Kempe chain beginning at the boundary region clockwise to the vertex and whose other color is that of the region two spaces counterclockwise of the vertex (in Figure 1, B and Y) is called the *left-hand chain.*

- A Kempe chain beginning at the boundary region counterclockwise to the vertex and whose other color is that of the region two spaces clockwise of the vertex (in Figure 1, B and G) is called the *right-hand chain.*

- The Kempe chain containing the two boundary regions not adjacent to the vertex is called the *end tangent chain.* In this case, this would be a GY Kempe chain.

Using these definitions, we will state some of Kittell's operations.

- α : Exchanging colors on the left-hand chain

- β : Exchanging colors on the right-hand chain
- γ : Exchanging colors on the left-hand circuit (or the broken left-hand circuit)
- δ : Exchanging colors on the right-hand circuit (or the broken right-hand circuit)
- ϵ : Exchanging colors on the end tangent chain

Given a partial coloring c from a subset of the regions of a graph to the colors $\{R, B, Y, G\}$ and some operation σ, we will refer to $\sigma(c)$ as the resulting coloring after applying σ to c. In addition, given two operations σ_1, σ_2, we will refer to $\sigma_2\sigma_1$ as the result of applying first σ_1, then σ_2.

In Kittell's work, it was assumed that every coloring was at *impasse* prior to using these operations, an assumption we will not necessarily make. Traditionally, *impasse* means that the left-hand circuit and right-hand circuit "cross." However, since the manner of crossing is somewhat specific, we will instead define *impasse* to mean that $c, \alpha(c)$, and $\beta(c)$ each have both a left-hand and right-hand circuit. This is slightly stronger than the traditional understanding of impasse, but it will simplify our work. Indeed, if a coloring does not have a left-hand or right-hand circuit, then the operations γ and δ respectively result in only 3 colors amongs the boundary regions, allowing us to color the exterior with the remaining color. For practical reasons, it will be assumed that α is only applied to colorings having a left-hand circuit, and similarly β is only applied to colorings having a right-hand circuit. This ensures by planarity that the left-hand chain (and right-hand chain respectivley) only contains one boundary region.

Given these definitions and notation, let us state Errera's results from [2] more formally.

Proposition 2.1. *Let c_0 be the partial coloring of the Errera map in Figure 1. Then $\alpha^n(c_0)$ is at impasse for all $n \in \mathbb{N}$, and $\alpha^{20}(c_0) = c_0$.*

It is also readily observed that if c is at impasse, then $\alpha\beta(c) = \beta\alpha(c) = c$. This was noted in [7].

3 Impasse Colorings of Errera Map

We will begin by showing that only four colorings of the Errera map are impasse colorings (up to rotation and permutation of colors).

In Figure 2, we have numbered the interior regions of the Errera map. We have also assigned colors to the five regions adjacent to the exterior uncolored region. This can be done without loss of generality, as we can freely rotate the Errera map and permute colors until they are in the configuration shown in Figure 2. It bears mentioning that each of our four colorings of the interior region represents the $4! \cdot 5 = 120$ different colorings that can be obtained by permutation and rotation.

We begin with two cases: The regions 1 and 2 are colored Y and G respectively, or the regions 1 and 2 are colored G and Y respectively.

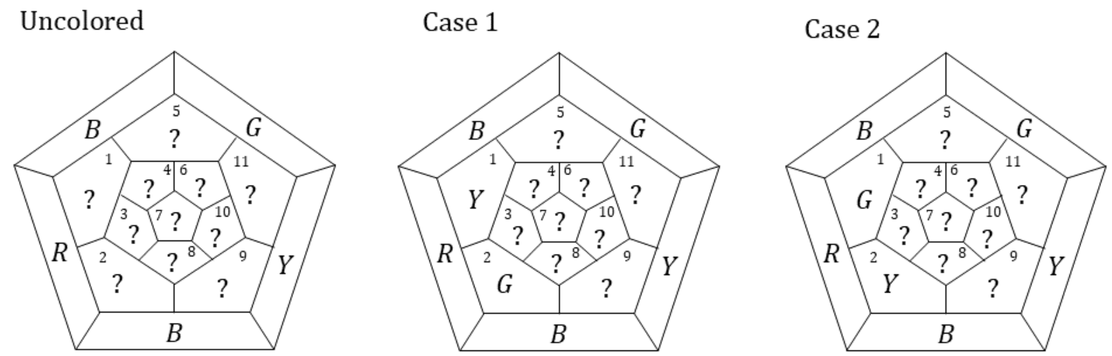

Figure 2: Blank numbered copy of the Errera map and our two primary cases.

We begin with a general observation that will assist in narrowing down possible impasse colorings.

Observation 3.1. *Neither region* 6 *nor region* 10 *can be colored* R.

Proof. In an impasse coloring, there must be a RY circuit. This means that region 9 or region 11 must be colored R. Since region 10 is adjacent to both region 9 and region 11, it may not be colored R. A similar argument except with the RG circuit shows that region 6 cannot be colored R. □

We now proceed to our cases.

3.1 Case 1

In this case, regions 1 and 2 are colored Y and G respectively. We now consider the location of the next R region in the RY circuit. Before we do, let us observe a fact that applies to all colorings in Case 1.

Observation 3.2. *In Case 1, neither region* 4 *nor region* 8 *can be colored* R.

Proof. If region 4 were colored R, then all four colors would be adjacent to region 5, and thus region 5 could not be colored. This also applies to region 8. □

Therefore, we see that the next region in the RY circuit must be either region 5 or region 3. These will be Cases 1.1 and 1.2, as shown in Figure 3

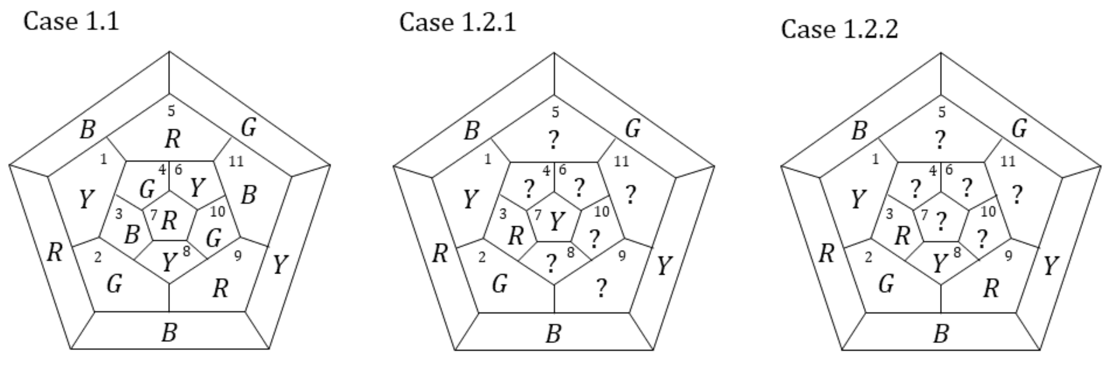

Figure 3: Colorings for Cases 1.1, 1.2.1, and 1.2.2.

In Figure 3, the remainder of the graph for Case 1.1 is colored as well. Starting with just regions 1, 2, and 5 colored, we see that the only region adjacent to 5 that is not adjacent to another Y region is 6. Due to the R in region 5 and Observation 3.1, the next R region must be region 7. The only region adjacent to 7 that is not also adjacent to a Y is 8, which must be the next region in the RY circuit. Similarly, the only region adjacent to 8 that is not adjacent to a R is region 9. This completes the RY circuit, and regions $1, 2, 5, 6, 7, 8$, and 9 have been colored. Region 3 is now adjacent to three different colors and must therefore be colored the fourth color, B. The same applies to region 11, and then also to regions 4 and 10 which must be colored G. Thus, this starting coloring of regions $1, 2$, and 5 determines the remainder of the coloring.

In Case 1.2, there are once again two different places where the next region of the RY circuit may be located: region 7 or region 8. These are divided into Cases 1.2.1 and 1.2.2. One can quickly deduce by Observations 3.1 and 3.2 that Case 1.2.1 cannot lead to an impasse coloring of the Errera map. Therefore, we turn our attention to Case 1.2.2. Note that region 9 has been colored R, due to the R in region 3 and Observation 3.1. This completes the RY circuit. Now we turn our attention to the RG circuit. The next G region in the RG circuit may be placed in either region 4 or 7, giving cases 1.2.2.1 and 1.2.2.2 depicted in Figure 4.

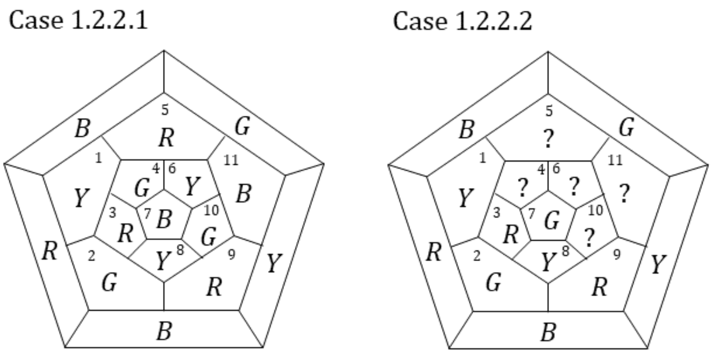

Figure 4: Colorings for Cases 1.2.2.1 and 1.2.2.2

Once again, an entire coloring for Case 1.2.2.1 has been provided. Given colorings of regions $1, 2, 3, 4, 8$, and 9, we see that region 5 is adjacent to three different colors (and in fact, has been in all of Case 1) and must be colored R. Region 7 is also adjacent to three colors, and therefore must be colored B. Now region 6 is adjacent to three colors and must be colored Y, then similarly region 11 must be colored B, and region 10 must be colored G. Thus, the remainder of the coloring is determined.

In Case 1.2.2.2, we deduce as in Case 1.2.1 by Observations 3.1 and 3.2 that we cannot have an impasse coloring of the Errera map. Thus, we have completed our examination of Case 1, resulting in only two potential impasse colorings of the Errera map.

3.2 Case 2

In this case, regions 1 and 2 are colored G and Y respectively. Again, we try to complete the RY circuit. First, we make an observation slightly different than those preceding.

Observation 3.3. *In Case 2, neither region* 5 *nor region* 9 *can be colored* R.

Proof. Let c be an impasse coloring in Case 2. Note that since c is in impasse, there is an RY circuit, and therefore there is no Kempe chain from the B boundary region counterclockwise to the vertex to the G boundary region. If region 5 was colored G in a coloring c, then the left-hand chain (a BY chain) would consist of only the boundary region clockwise to the vertex. Therefore, $\alpha(c)$ would have the G boundary region as its vertex, and there would still be an RY-chain stretching from the R boundary region to the Y boundary region clockwise to the vertex. Therefore, $\alpha(c)$ could not have a left-hand circuit (which in this case is a BG circuit), which contradicts that c is at impasse. The case for region 9 is similar. □

With this result in hand, we attempt to continue the RY circuit. Considering Observation 3.3, the next R region can either be in region 3 or 8. These are Cases 2.1 and 2.2 in Figure 5, with Case 2.1 divided into two smaller cases.

Neither Case 2.1.1 not Case 2.1.2 can lead to an impasse coloring of the Errera map. In Case 2.1.1, the R in region 3 and Observations 3.1 and 3.3 eliminate all options for the next R region in the RY circuit. Similarly, in Case 2.1.2 the R in region 3 and Observation 3.1 eliminate all options for the next R region. Therefore, we turn our attention to Case

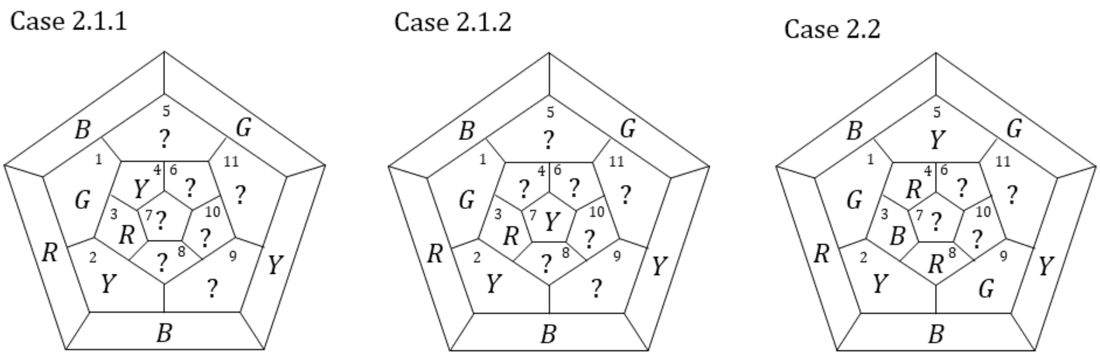

Figure 5: Colorings for Cases 2.1 and 2.2

2.2. Once again, some additional regions have been colored. Given colorings of regions $1, 2,$ and 8, we see that regions 3 and 9 are adjacent to three colors and must be colored B and G respectively. We also note that the next R in the RG circuit cannot be in region 5 by Observation 3.3, and thus must be in region 4. Now region 5 is also adjacent to three colors and must be colored Y.

The next Y region of the RY circuit can then be in either region 7 or 10. These will be Cases 2.2.1 and 2.2.2 in Figure 6. In each case, a complete coloring of the interior regions of the Errera map is determined.

In Case 2.2.1, we start with a coloring of regions $1, 2, 3, 4, 5, 7, 8,$ and 9. Then region 10 is now adjacent to three colors and must be colored B. This then causes regions 6 and 11 each to be adjacent to three colors, and therefore they must be colored G and R respectively. Therefore a coloring has been determined. In Case 2.2.2, we again start with a coloring of regions $1, 2, 3, 4, 5, 7, 8,$ and 9. This time, region 6 is adjacent to three colors and must be colored B. This then causes regions 10 and 11 to be adjacent to three colors, and therefore they must be colored Y and R respectively.

Thus we have completed our analysis of cases. The only cases which potentially provide impasse colorings of the Errera map are Cases 1.1, 1.2.2.1, 2.2.1, and 2.2.2. Note that the coloring in Case 1.2.2.1 is the same as that in Figure 1. We present again the remaining

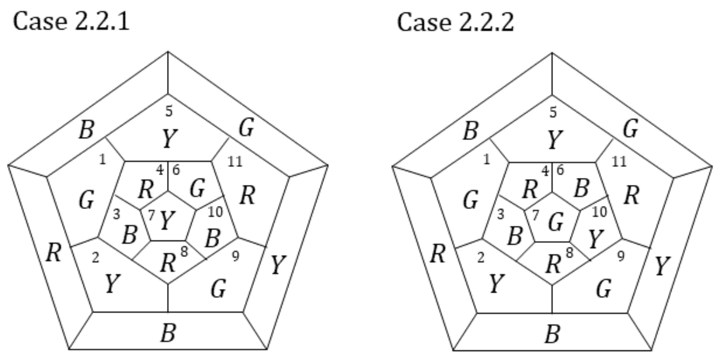

Figure 6: Colorings for Cases 2.2.1 and 2.2.2

three colorings in Figure 7.

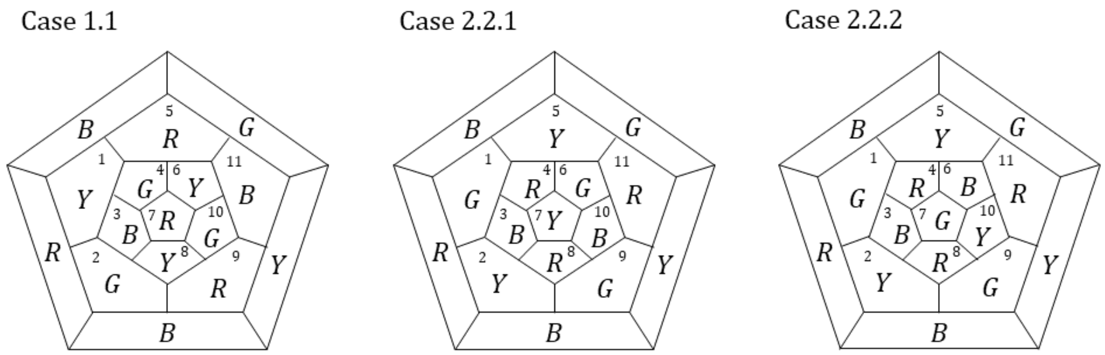

Figure 7: Three potential impasse colorings of the Errera map

4 Verifying Cyclicity

Thus far, we have narrowed our search to four colorings of the interior regions of the Errera map which may have impasse colorings. We have neither shown that these are impasse colorings, let alone colorings having the same cyclic pattern as the coloring c_0 in Figure 1.

However, we are aided by the following observation:

Observation 4.1. *Let c be a coloring of the interior regions of the Errera map such that α^n is at impasse for all n, and $\alpha^{20}(c) = c$. Then this is also true for all colorings $\alpha^k(c)$, for any k.*

If we can show that each of the colorings in Figure 7 is $\alpha^k(c_0)$ for some k, perhaps under some rotation and permutation of colors, then this will show that the coloring is at impasse and has the same cyclic pattern as the original Errera map.

First, we apply α to the coloring c_0 in Figure 1 and permute the colors by $(GRYB)$. This is shown in Figure 8. We note that this is a rotation of the coloring in Case 2.2.2. Therefore, the coloring in Case 2.2.2 is equivalent to $\alpha(c_0)$.

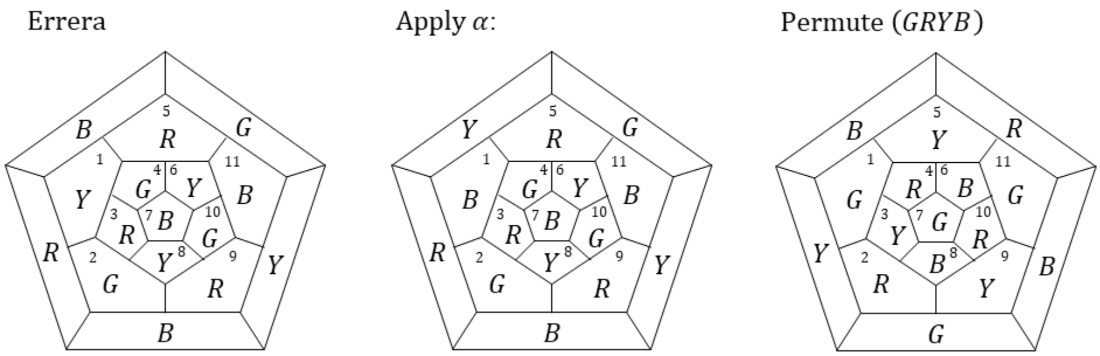

Figure 8: Original coloring under α, then under permutation of colors. Results in a rotation of Case 2.2.2.

We will now applying α^2 and α^3 to c_0, permuting colors appropriately. These results are given in Figure 9, with the colorings being rotations of Cases 1.1 and 2.2.1 respectively.

We have now described all impasse colorings of the Errera map, which happen to also be colorings leading to a cyclic pattern. This has of course assumed that the uncolored region is one of the pentagonal regions adjacent to no hexagons. Some pentagonal regions of the Errera map are adjacent to hexagons, although these will not be studied here.

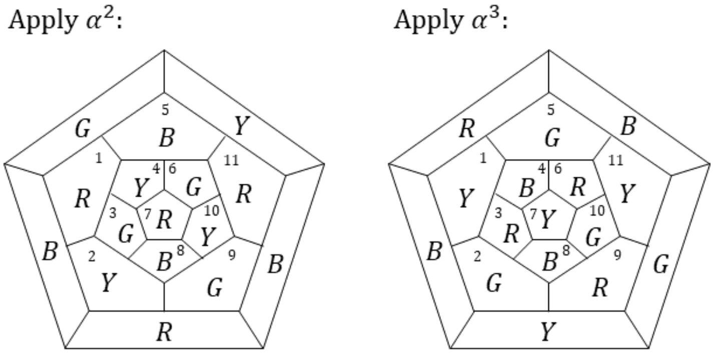

Figure 9: Original coloring under α^2 and α^3, then under permutation of colors. Results in rotations of Case 1.1 and Case 2.2.1 respectively.

We can neatly summarize our findings in the theorems below. For the sake of brevity here and later, we will say that R is a *central pentagon* of the Errera map if it is a pentagonal region whose neighbors are also pentagonal regions. We will assume in the following that the Errera map is drawn so that the exterior region is a central pentagon.

Theorem 4.2. *Let G be the Errera map drawn so that the exterior region is a central pentagon, let c be a proper coloring of the interior regions of G, and let c_0 be the coloring in Figure 1. Then the following are equivalent:*

1. *c is an impasse coloring.*
2. *c is equivalent to $c_0, \alpha(c_0), \alpha^2(c_0)$, or $\alpha^3(c_0)$, possibly under some rotation and/or permuation of colors.*
3. *$\alpha^n(c)$ is at impasse for all n, with $\alpha^{20}(c) = c$.*

Corollary 4.3. *Let G be the Errera map drawn so that the exterior region is a central pentagon. There are exactly* 480 *proper colorings of the interior regions of G that are at impasse, counting rotations and permutations as separate colorings.*

We end this section by presenting in Figure 10 all four impasse colorings of the Errera map (up to rotation and permutation of colors). Note that the interior central pentagon of each is colored differently, which makes it easier to distinguish which coloring is present in a given instance.

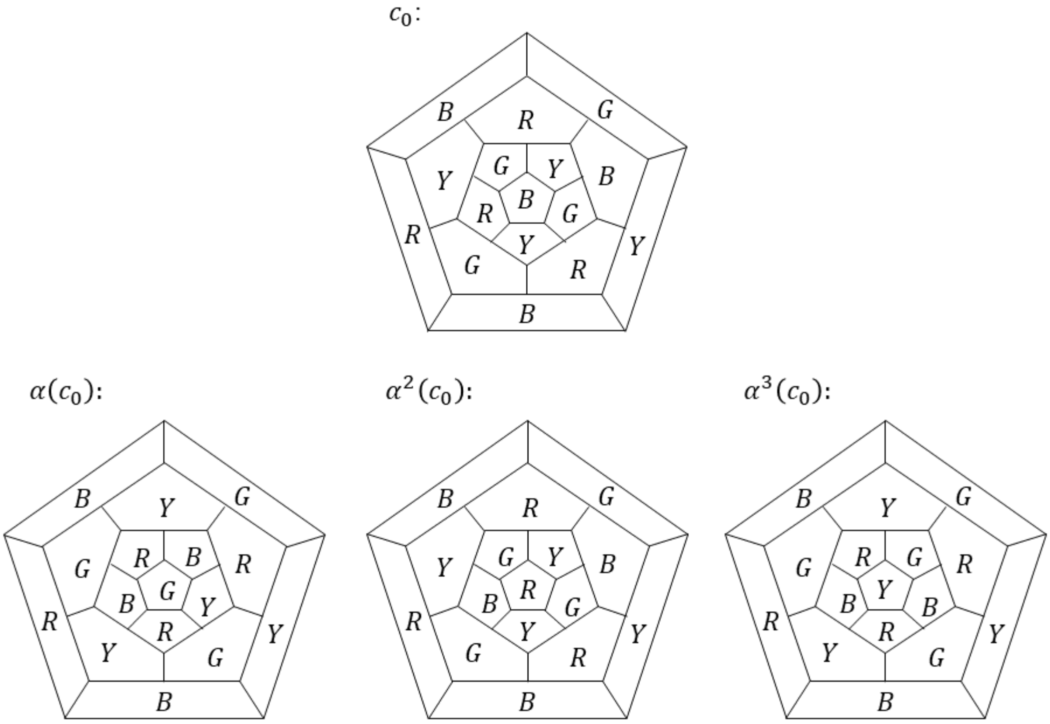

Figure 10: All four impasse colorings of the Errera map.

5 Resolving Impasses

Now that we have succinctly described all possible impasse colorings of the interior regions of the Errera map, we can briefly discuss how to resolve impasses in all such cases.

The following observation is made by Kittell in [7].

14

Observation 5.1. *Let c_0 be the coloring of the Errera map in Figure 1. Then $\epsilon(c_0)$ is not at impasse.*

Figure 11 demonstrates this to be true. While there are certainly some paths that appear to cross, the existence of at least one RY circuit and one RG circuit that do not cross is enough to demonstrate that the coloring is not at impasse by the traditional definition. By our revised definition, we note that $\alpha\epsilon(c_0)$ does not have a left-hand circuit, and therefore $\gamma\alpha\epsilon(c_0)$ has only three colors in boundary regions.

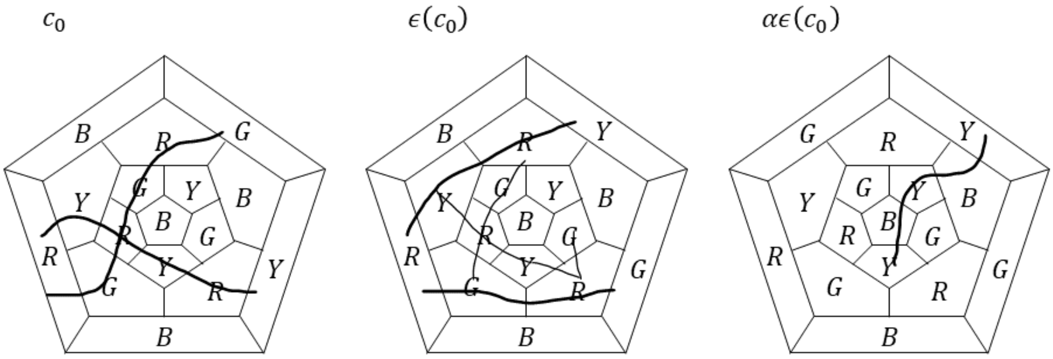

Figure 11: Resolution of impasse in c_0 by ϵ.

One may notice that c_0 and $\alpha^2(c_0)$ are very similar. In fact, we can observe the following (leaving verification to the reader).

Observation 5.2. *Let c_0 be the coloring of the Errera map in Figure 1. Then $\epsilon(\alpha^2(c_0))$ is not at impasse.*

One more observation of Kittell's will be useful.

Observation 5.3. *Let c_0 be the coloring of the Errera map in Figure 1. Then $\alpha^4(c_0)$ is a rotation of c_0.*

This should not be surprising, as $c_0, \alpha(c_0), \alpha^2(c_0)$, and $\alpha^3(c_0)$ make up all the distinct impasse colorings of the Errera map. Given our characterization of the impasse colorings of the Errera map and these observations, we can now characterize resolutions of impasse in the Errera map.

Theorem 5.4. *Let G be the Errera map drawn so that the exterior region is a central pentagon, and let c be a proper coloring of the interior regions of G which is at impasse. Then the impasse can be resolved by either ϵ or $\epsilon\alpha$.*

Proof. By Theorem 4.2 we may assume that c is $\alpha^n(c_0)$ for $0 \leq n \leq 3$. By Observations 5.1 and 5.2 we have handled the cases $n = 0, 2$. For $n = 1, 3$, note that applying $\epsilon\alpha$ to $\alpha(c_0)$ and $\alpha^3(c_0)$ results in $\epsilon\alpha\alpha(c_0) = \epsilon\alpha^2(c_0)$ and $\epsilon\alpha\alpha^3(c_0) \simeq \epsilon(c_0)$, neither of which is at impasse. Thus we have handled all cases. □

6 Coloring a Graph Containing the Errera Map

Suppose we are four-coloring the regions of a cubic map one-by-one, and upon attempting to assign a color to a region we find it has five colored neighbors using all four colors. It is possible that this graph contains a cycle C such that the map induced by all regions interior to the cycle is the Errera map (in a way which will be more precisely defined later), with the region currently trying to be colored corresponding to a central pentagon. In such a case, can the findings above translate to a method to assign a color to this region?

First, let us rephrase this condition in a few ways. First, we may as well exchange "inside" with "outside," so that all of the regions not belonging to the Errera map are interior to the cycle C. Next, we will say a cubic map is the Errera map *with a hole* if there is a cycle C containing any number of regions such that the graph formed by contracting the cycle and all regions interior of the cycle to a vertex is isomorphic to either the Errera map or a subdivision of the Errera map. We may also allow the Errera map to have more than one hole. Examples of Errera maps with holes, as well as cycles which would not count as holes in this sense, are depicted in Figure 12. Note that the dual of an Errera map with holes either contains the Errera map dual as an induced subgraph, or otherwise is a multigraph containing the dual of the Errera map with added parallel edges as an induced subgraph.

16

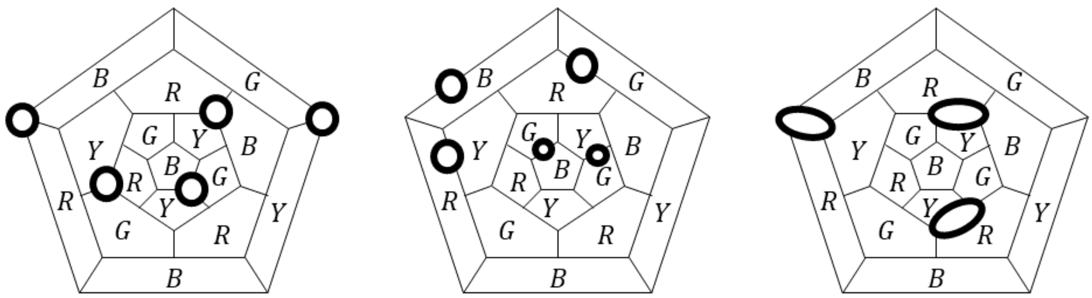

Figure 12: The first two graphs are the Errera map with "holes" as described above. The bolded circles may contain any number of regions. Contracting the cycles in the third graph results in vertices of degree 4 or more, and thus the result is not the Errera map or a subdivision of the Errera map.

So how many of the results of the previous section translate to the Errera map with holes? The answer is nearly all of them.

Consider a partial coloring of an Errera map with holes. Suppose we would like to assign a color to an uncolored region with at most five colored regions which, after contraction of all holes and vertices of degree 2, is a central pentagon of the Errera map. For brevity, we will refer to such a region simply as a *central pentagon*, even though it or its neighbors may not be pentagons prior to contraction. Further assume that this central pentagon is drawn as the exterior region. Assume also that the only colored neighbors of the exterior region are those on the Errera map proper (that is, no colored region of a hole touches the exterior region). We will make the following observations, with justification where necessary.

Observation 6.1. *Adding holes preserves adjacency relations of the Errera map.*

Observation 6.2. *A proper partial coloring of the Errera map with holes induces a proper partial coloring of the Errera map.*

The following fact takes some justification.

Proposition 6.3. *The AB Kempe chains of a partially colored Errera map with holes containing at least one region of the Errera map induce AB Kempe chains on the induced partial coloring of the Errera map, and vice versa.*

Proof. Let $\mathcal{K}$ be the collection of Kempe chains of the Errera map with holes containing at least one region of the Errera map, and let $\mathcal{K}'$ be the collection of Kempe chains of the Errera map. First, note that the partial coloring of the Errera map with holes induces a proper partial coloring on the Errera map. Therefore, a Kempe chain $K \in \mathcal{K}$ contains one or more Kempe chains $K'_1, K'_2, ...K'_t \in \mathcal{K}'$. We will show K contains only a single such Kempe chain.

Suppose to the contrary that there are $t \geq 2$ Kempe chains of the Errera map contained in K as described above. Then the regions connecting these Kempe chains must belong to holes. Let H be a hole connecting at least two distinct Kempe chains K'_i and K'_j. Then there are regions R'_i and R'_j of K'_i and K'_j respectively that are both adjacent to regions inside H. Note that all regions bordering H are mutually adjacent in the Errera map. Therefore, since R'_i, R'_j are both bordering H, they must also be adjacent in the Errera map. Thus the Kempe chains K'_i and K'_j are connected. This is a contradiction, and the Kempe chain $K \in \mathcal{K}$ induces exaclty one Kempe chain $K' \in \mathcal{K}'$.

Thus we have defined a map $f : \mathcal{K} \to \mathcal{K}'$ from a Kempe chain in the Errera map with holes to its induced Kempe chain in the Errera map. Since every Kempe chain $K' \in \mathcal{K}'$ is a subgraph of exactly one Kempe chain $K \in \mathcal{K}$ (certainly not multiple disconnected Kempe chains, as they must share all regions of K' in common, contradicting disconnectivity), we also have an inverse $f^{-1} : \mathcal{K}' \to \mathcal{K}$. Thus, we have shown a bijection between the $\mathcal{K}$ and $\mathcal{K}'$. □

Next we would like to make an observation about performing Kittell's color exchange operations on a partially colored Errera map with holes. However, we will now need to modify our definitions slightly to accomodate additional uncolored regions. First, we will always be interested in attempting to assign a color to a specific uncolored region R, which has at most 5 colored neighbors, all belonging to the Errera map. We will assume that R is the exterior region, and R will take the place of the single uncolored region in our previous definitions. Also, when counting regions clockwise and counterclockwise from a given colored boundary region, we ignore any uncolored regions. If the boundary of a region is interrupted by a hole, as in the second graph in Figure 12, then we ignore the (by necessity uncolored) boundary regions in the hole and count the interrupted region as a single region. Lastly, it might be in the general case that the boundary regions having the same color are consecutive but separated by uncolored regions. Since adjacency is preserved by adding holes this will not be the case here; however, a resolution to such a situation is provided in [3].

We next observe the following, with proof.

Observation 6.4. *Let c be a partial coloring of an Errera map with holes, let c' be the induced coloring of the Errera map, and let σ be a color exchange operation. Then the proper coloring $\sigma(c)$ induces the proper coloring $\sigma(c')$ on the Errera map.*

Proof. Let K be a Kempe chain in the Errera map with holes that begins at a colored boundary region. Such a region necessarily belongs to the Errera map, and thus this induces a Kempe chain K' on the Errera map beginning at that same region. Then, $\sigma(c)$ exchanges the colors on K. This induces a proper coloring of the Errera map without holes, where the colors on K' have been exchanged. This is exactly $\sigma(c')$. □

These observations culminate in the following theorem about impasse in the Errera map with holes.

Theorem 6.5. *A partial coloring of the Errera map with holes is at impasse if and only if the induced coloring of the Errera map is also at impasse.*

Proof. Let c be a partial coloring of the Errera map with holes, and c' be the induced partial coloring of the Errera map. First, suppose c is at impasse. Then c has a left-hand circuit and right-hand circuit. Since these are Kempe chains starting at colored boundary regions, and all colored boundary regions are also boundary regions of the Errera map, this induces left-hand and right-hand circuits in c'. If c is at impasse, $\alpha(c)$ and $\beta(c)$ each have a left-hand circuit and a right-hand circuit. These induce colorings which also have left-hand and right-hand circuits; by Observation 6.4, these colorings are exactly $\alpha(c')$ and $\beta(c')$. Therefore c' is at impasse.

For the reverse direction, suppose that c' is at impasse. Then $\alpha(c)$ and $\beta(c)$ induce colorings $\alpha(c')$ and $\beta(c')$. Since c' is at impasse, $c', \alpha(c')$, and $\beta(c')$ each have left-hand and right-hand circuits. These circuits are each subgraphs of left-hand and right-hand circuits in $c, \alpha(c)$, and $\beta(c)$ respectively. Thus, c is also at impasse. □

As a result of Theorem 6.5, most of our previous results follow. If the Errera map with holes is at impasse, then the induced coloring of the Errera map is also at impasse. Applying α any number of times will still result in an impasse coloring of the Errera map, which translates to an impasse coloring of the Errera map with holes. However, while applying α^{20} returns the coloring of the Errera map to its original impasse coloring, the same is not necessarily true of regions inside holes. However, since the number of colorings is finite and applying α repeatedly does not resolve the impasse, it follows that the sequence of partial colorings generated by α will also be cyclic with some period p such that $20|p$.

Of most interest is the fact that, since we can resolve all impasse colorings of the Errera map, we can also resolve all impasse colorings of the Errera map with holes.

Theorem 6.6. *Let G be an Errera map with holes, and let c be a partial coloring of G, where all regions of the Errera map except for a central pentagon R have been assigned a color. Suppose further that the only colored neighbors of R belong to the Errera map, and when attempting to assign a color to R, the coloring c is at impasse. Then the impasse can be resolved by either ϵ or $\epsilon\alpha$.*

7 An Example, a Motivation, and a New Example

We begin by exhibiting in Figure 13 an example of an Errera map with a hole. This graph was generated randomly using the reduction algorithm described by Morgenstern and Shapiro in [8]. Specifically, this was generated as a maximal planar graph, then converted to a cubic map. The regions of the map have been carefully arranged to make its structural properties apparent.

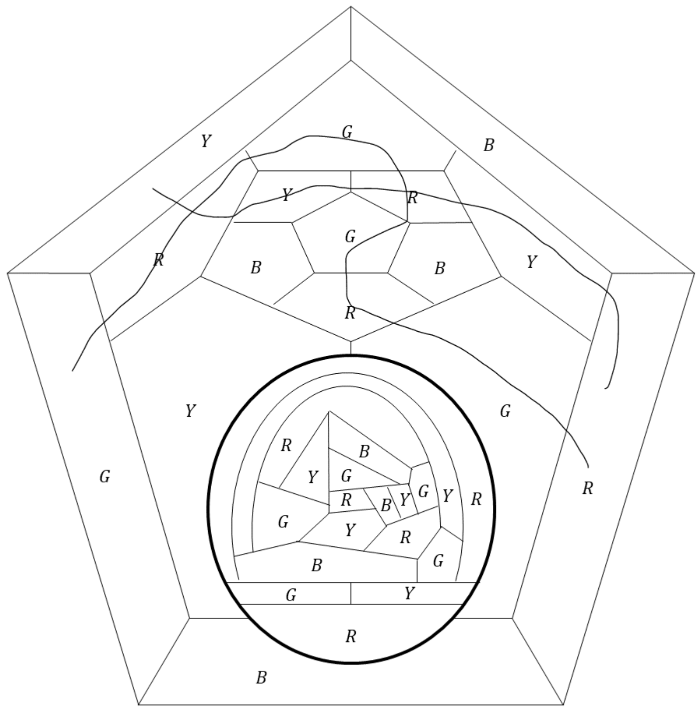

Figure 13: An example of an impasse coloring of an Errera map with a hole.

We will note that it takes 60 applications of α to return this graph to its original coloring (leaving verification to the reader). We can also observe by the fact that the interior central pentagon is colored G that the induced coloring of the Errera map is a rotation of $\alpha(c_0)$. Therefore, we should be able to resolve the impasse by $\epsilon\alpha$ and furthermore obtain only three colors on the boundary regions by next applying $\gamma\alpha$. In Figure 14 we show the coloring obtained by applying $\gamma\alpha\epsilon\alpha$ to the map in Figure 13. At this point, the exterior region can be colored R, obtaining a coloring of all regions of the map.

One may wonder why Errera maps with holes are of particular interest. On one hand, we have seen that certain partial colorings of Errera maps with holes are at impasse under repeated application of α, showing that this entire class of graphs is rather troublesome. On the other hand, we have now developed a method of resolving impasses in these troublesome cases with only a handful of color exchanges.

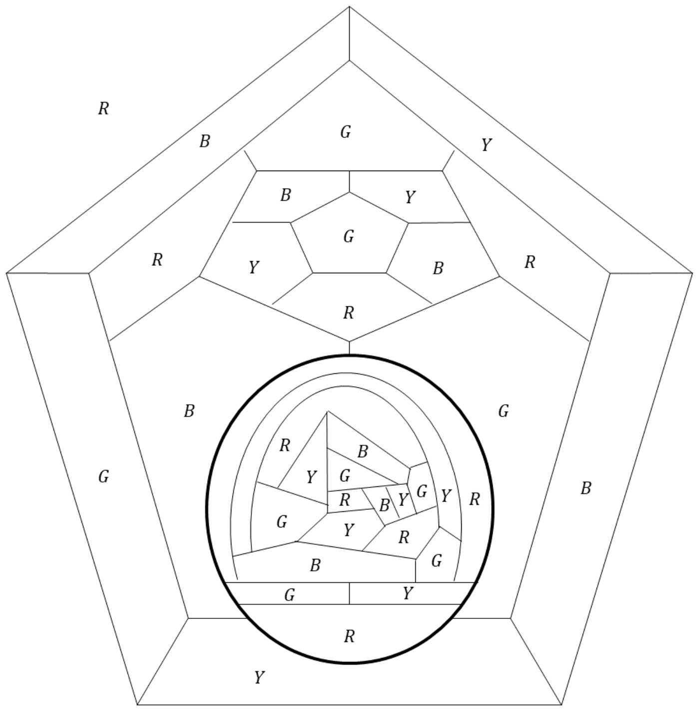

Figure 14: Coloring of Errera map with a hole after applying $\gamma\alpha\epsilon\alpha$.

Original experimental data suggested that the Errera maps with holes were the only graphs for which α failed to resolve impasse. However, additional tests have revealed some exceptions. One such graph G is shown in Figure 15; we will call the coloring c.

In this case, while we see that the Errera map makes an appearance, some adjacency relations between regions are not maintained. This represents missing edges in the dual graph. Like the example in Figure 13, we see that $\alpha^{60}(c) = c$. Interestingly, one can observe that considering the regions that correlate to regions of the Errera map, the colorings $\alpha^k(c)$ restricting to the Errera map are each impasse colorings of the Errera map. Unlike the case with the Errera map with holes where this can be proven to happen, this seems to be purely a remarkable coincidence. It remains to be seen if there are certain properties of the map G that lead to this occurrence.

We end by observing that, in many impasse colorings c of cubic maps G, α^n resolves

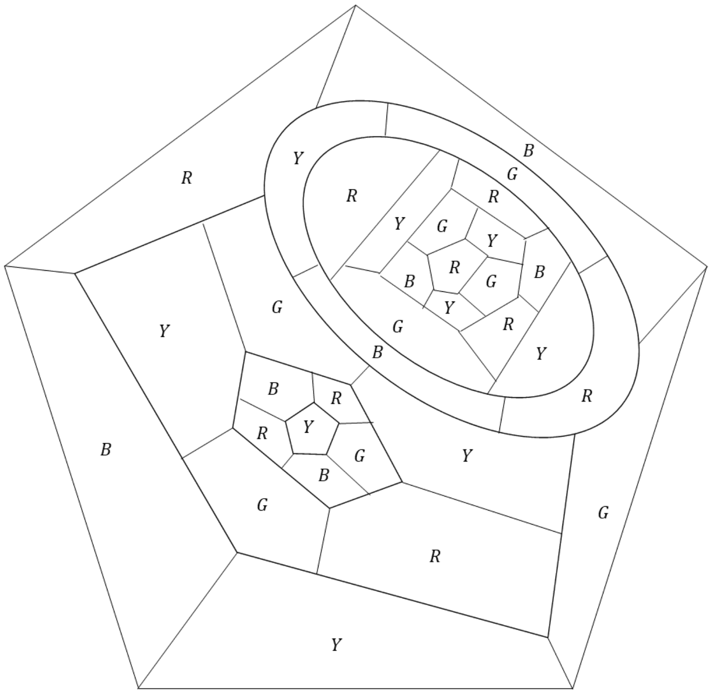

Figure 15: An example of an impasse coloring of a map that is not an Errera map with holes as defined above, as not all adjacency relations are maintained.

impasse for some n. In the cases where c is an impasse coloring of the Errera map or an Errera map with holes, then we have proven that ϵ or $\epsilon\alpha$ resolves impasse. It has also been seen experimentally that in other cases where α^n does not resolve impasse that $\epsilon\alpha^n$ still resolves impasse. This leads to a final question:

Question 7.1. *Can it be shown that for any impasse coloring c of a cubic map G that either α^n or $\epsilon\alpha^n$ resolves impasse for some $n \in \mathbb{N}$?*

Data Availability Statements The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations On behalf of all authors, the corresponding author states that there is no conflict of interest.

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1

On Finite Rings and Their Group of Units

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The purpose of this paper is to review old results on multiplicative groups of units of finite rings and also correct some erroneous results.

Let R denote a ring with identity. The Jacobson radical of R is denoted by $J = J(R)$. The multiplicative group of units in R is denoted by $R^\times$. The Galois field of order q is denoted by $GF(q)$. The ring of $n \times n$ matrices over R is denoted by $M_n(R)$ and the general linear group of degree n over R is denoted by $GL_n(R)$. For a ring R and a non-negative integer k, $R^{(k)}$ denotes the direct sum of k's copies of R. For a finite set S, we denote the number of elements in S by $|S|$.

Basis results on finite rings

A division ring is a nontrivial ring in which every nonzero element a has a multiplicative inverse, that is, an element usually denoted a^{-1}, such that $aa^{-1} = a^{-1}a = 1$.

For example, $\mathbb{H} = \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \right\} \subset M_2(\mathbb{C})$ is a noncommutative division ring.

However, for finite rings, there is no distinction between domains, division rings and fields.

Theorem 1. (Wedderburn [**?**]) *A finite division ring is a field.*

[1]The detailed version of this paper will be submitted for publication elsewhere.

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A right Artinian simple ring is isomorphic to a matix ring $M_n(D)$ for some positive integer n and some division ring D. Hence, by Wedderburn's little thore we obtain

Theorem 2. *A finite simple ring is isomorphic to a matix ring $M_n(F)$ for some positive integer n and some finite field F.*

Theorem 3. *A finite ring R has a unique maximal nilpotent ideal J and R/J is a finite direct sum of simple rings. We call J the Jacobson radical of R.*

The book "Finite Rings With Identity" by Bernard R. McDonald [?] is a good reference for finite rings.

Some results on the group of units of a finite ring

The following are some examples of groups of units of finite rings.

Example 4. *Consider the ring $R = \mathbb{Z}/8\mathbb{Z} = \{\bar{0},\ \bar{1},\ \bar{2},\ \bar{3},\ \bar{4},\ \bar{5},\ \bar{6},\ \bar{7}\}$. Then $R^\times = \{\bar{1},\ \bar{3},\ \bar{5},\ \bar{7}\}$. This is the Klein four-group, that is an abelian group with four elements, in which each element is self-inverse.*

Example 5. *Consider the ring $R = M_2(\mathbb{Z}/2\mathbb{Z})$. Then $R^\times = GL_2(\mathbb{Z}/2\mathbb{Z})$*

$$= \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & 0 \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} 0 & \bar{1} \\ \bar{1} & 0 \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\} \cong S_3.$$

Example 6. *Let G be any finite group and let F be any finite field. Consider the group ring FG of G over F. Then $G \subset (FG)^\times$. Hence any finite group is a subgroup of some group of units of a finte ring.*

For a general ring R with identity, the **Jacobson radical** $J = J(R)$ of a ring R is the two-sided ideal of R defined by

$$\begin{aligned} J =& \text{the intersection of all maximal right ideals of the ring } R \\ =& \text{the intersection of all maximal left ideals of the ring } R. \end{aligned}$$

The Jacobson radical J of a ring R is characterized by the following:

$$J = \{\ x \mid 1 + rx \text{ is a unit for all } r \in R\}.$$

Proposition 7. $1+J$ *is a normal subgroup of* $R^\times$ *and* $R^\times/(1+J) \cong (R/J)^\times$.

Proof. From this fact, we see that $1+J$ is a normal subgroup of $R^\times$. Consider the canonical ring epimorphism $f : R \to R/J$, that is, $f(a) = a+J$ for all $a \in R$. Suppose that $a+J$ is a unit in R/J. Then there is an element $b \in R$ such that $ab+J = ba+J = 1+J$. Then $ab = 1+x$ and $ba = 1+y$ for some $x, y \in J$ and hence $ab(1+x)^{-1} = (1+y)^{-1}ba = 1$. Therefore a is a unit of R. Hence f induces the group epimorphism $R^\times \to (R/J)^\times; a \to a+J$. Since $\mathrm{Ker}(f) = 1+J$, we have $R^\times/(1+J) \cong (R/J)^\times$.

For the rest of this section, R denotes a finite ring with Jacobson radical J.

Let G be a group. Recall that the commutator of $x,\ y \in G$ is $[x, y] = xyx^{-1}y^{-1}$. A group G is **nilpotent** if there is an integer $m > 1$ such that $[\dots[g_1, g_2], g_3] \dots, g_m] = 1$ for all $g_1,\ g_2, \cdots, g_m \in G$.

Proposition 8. $1+J$ *is a nilpotent subgroup of* $R^\times$.

Proof. Let n be the least positive integer such that $J^n = 0$. Then $1+J \supset 1+J^2 \supset \cdots \supset 1+J^{n-1} \supset 1$ is a central series of $1+J$.

Theorem 9. (Eldridge [?]) *Let* R *be a finite ring. Then* $R^\times$ *is a nilpotent group if and only if* R/J *is a direct sum of fields.*

Proof. There is a exact sequence of groups:

$$1 \to 1+J \to R^\times \to (R/J)^\times \to 1.$$

Since $1+J$ is nilpotent, $R^\times$ is a nilpotent group if and only if $(R/J)^\times$ is nilpotent. For some positive integers n_k and some finite fields F_k, we have $R/J = \bigoplus_{i=1}^m M_{n_k}(F_k)$ and so $(R/J)^\times = \prod_{i=1}^m GL_{n_k}(F_k)$. Hence, $(R/J)^\times$ is nilpotent if and only if R/J is a direct sum of fields.

Remark 10. *For a right Artinian ring* R*, there is also a exact sequence of groups:*

$$1 \to 1+J \to R^\times \to (R/J)^\times \to 1$$

In this case, R/J *is a finite direct sum of matrix rings over some division rings. Hence we have the following:*

Theorem 11. *Let R be a right Artinian ring. Then $R^\times$ is finite if and only if R is a finite ring.*

However this result is not true for a right Noetherian ring, as the following example shows.

Example 12. *Consider the polynomial ring $R = GF(2)[X]$ over the Galois field $GF(2)$. We can see $R^\times = GF(2) = \{1\}$, but R is not a finite ring.*

On finite rings whose groups of units are solvable

The group $G' = G^{(1)}$ generated by the commutators $[x, y]$ in G is called the commutator or first derived subgroup of G. The second derived subgroup of G is $G^{(2)} = (G')'$; the third is $G^{(3)} = (G^{(2)})'$; and so on. A group G is **solvable** if and only if its kth derived subgroup $G^{(k)} = 1$ for some k.

Theorem 13. (Eldridge [**?**]) *Let R be a finite ring. Then $R^\times$ is solvable if and only if R/J is isomorphic to a finite direct sum of copies of the following rings:*
(i) $GF(p^n)$ where p is a prime and n is a positive integer,
(ii) $M_2(GF(2))$,
(iii) $M_2(GF(3))$.

Proof. (ii) $GL_2(GF(2)) \cong S_3$.

(iii) $GL_2(GF(3)) \supset SL_2(GF(3)) \supset Z$ are normal series, where Z is the center of $SL_2(GF(3))$, and
$SL_2(GF(3))/Z \cong PSL_2(GF(3)) \cong A_4$. It is well-known that A_4 has the Klein four-group V as a normal subgroup and A_4/V is a cyclic group of oder 3.

On finite rings whose groups of units are abelian

Lemma 14. *Let J be a finite local ring, that is R/J is a finite field. If $R^\times$ is abelian, then R is a commutative ring.*

Proof. Let a and b are two nonzero elements of R. If $a,\ b \in R^{\times}$, then $ab = ba$. If $a \in R^{\times}$ and $b \notin R^{\times}$, then $a,\ 1+b \in R^{\times}$ and hence $ab = ba$. If $a,\ b \notin R^{\times}$, then $1+a,\ 1+b \in R^{\times}$ and also $ab = ba$.

Proposition 15. *R is a finite noncommutative indecomposable ring. If $R^{\times}$ is abelian, then $R/J \cong GF(2)^{(n)}$ the direct of of n copies of $GF(2)$ for some $n > 1$.*

Proof. We know that $R/J = \bigoplus_{i=1}^{m} M_{n_k}(F_k)$ for some finite fields F_k. Since R is not commutative and $R^{\times}$ is abelian, we see that $m \geqq 2$ and $n_1 = \cdots = n_m = 1$, that is $R/J = F_1 \oplus \cdots \oplus F_m$ for some $m \geqq 2$. Let e and f be two orthogonal idempotents of R such that $\bar{e}$ is the identity of F_1 and $\bar{f}$ is theidentity of F_2. Clearly $(e+f)R(e+f)$ also has an abelian multiplicative group of units. So we may assume that $R/J = F_1 \oplus F_2$. Then $R = eRe \oplus eRf \oplus fRe \oplus fRf$ and $e + f = 1$. Also, we may assume that $eRf \neq 0$ because R is indecomposable.

For $a,\ a' \in eRf$ and $b' \in (fRf)^{\times}$, we have $(e+a+f)(e+a'+b') = (e+a'+b')(e+a+f)$, and hence, $e+a'+ab'+b' = e+a+a'+b'$. Multiplying the both sides by e from the left, we obtain $e+a'+ab' = e+a+a'$, and hence $a(b'-f) = 0$. If $F_2 = fRf/fJf$ has more than two elements, then there is an element b' in fRf such that $b'-f \in (fRf)^{\times}$. In this case, we have $a = 0$. But this implies $eRf = 0$, a contradiction. Therefore we conclude that $F_2 \cong GF(2)$. By similar way, we see that $F_1 \cong GF(2)$.

Example 16. *Let $n > 1$ and let*

$$R = \begin{pmatrix} GF(2) & GF(2) & \cdots & \cdots & GF(2) \\ 0 & GF(2) & GF(2) & \cdots & GF(2) \\ \cdots & 0 & GF(2) & \cdots & GF(2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & GF(2) & GF(2) \\ 0 & \cdots & \cdots & 0 & GF(2) \end{pmatrix},$$

the ring of $n \times n$ upper triangular matrices over $GF(2)$. Then the Jacobson radical of R is

$$J = \begin{pmatrix} 0 & GF(2) & \cdots & \cdots & GF(2) \\ 0 & 0 & GF(2) & \cdots & GF(2) \\ \cdots & 0 & 0 & \cdots & GF(2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & GF(2) \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

It can be esily seen that $T = R/J^2$ is a finite noncommutative indecomposable ring, $T^\times$ is abelian, and $T/J(T) \cong GF(2)^{(n)}$.

On finite rings whose groups of units are cyclic

A **Gilmer ring** is any finite ring whose multiplicative group of units is cyclic.

Theorem 17. (Ayoub [**?**], Gilmer [**?**], Eldridge and Fischer [**?**]), Raghavendran [**?**])
Each of the following rings is a Gilmer ring:
(a) $GF(p^n)$ where p is a prime and n is a positive integer,
(b) $\mathbb{Z}/(p^n)$ where p is an odd prime and $n \geqq 2$,
(c) $GF(p)[X]/(X^2)$ where p is a prime,
(d) $\mathbb{Z}/(4)$,
(e) $GF(2)[X]/(X^3)$,
(f) $\mathbb{Z}[X]/(4,\ 2X,\ X^2-2)$,
(g) $\begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$, the ring of upper triangular matrices over $GF(2)$.

Conversely, every indecomposable Gilmer ring is isomorphic to one of the rings described above.

Remark 18. *(a) $R = GF(p^n)$ is a finite field and $R^\times$ is a cyclic group of order $p^n - 1$. If $p \neq 2$, $p^n - 1$ is even.*
(b) Let $R = \mathbb{Z}/(p^n)$ where p is an odd prime and $n \geqq 2$. Then $|R^\times| = (p-1)p^{m-1}$ and this number is even.
(c) Let $R = GF(p)[X]/(X^2)$ where p is a prime. Then $|R^\times| = (p-1)p$ and this number is even.
(d) Let $R = \mathbb{Z}/(4)$. Then $R^\times = \{\bar{1}, \bar{3}\}$.
*(e) Let $R = GF(2)[X]/(X^3)$. Then $|R| = 8$ and $R^\times = 1 + J =$
$\{1,\ 1+\bar{X},\ 1+\bar{X}^2,\ 1+\bar{X}+\bar{X}^2\}$.*

(f) Let $R = \mathbb{Z}[X]/(4,\ 2X,\ X^2-2)$. *Then* $|R| = 8$ *and* $R^\times$
$= \{\bar{1},\ \bar{3},\ \bar{1}+\bar{X},\ \bar{3}+\bar{X}\} =<\bar{1}+\bar{X}> \cong GF(5)^\times$.
(g) Let $R = \begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$. *Then* $R^\times = 1 + J = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$.

By this remark, we have the following Table:

R	$R^\times$	$\|R\|$	$\|R^\times\|$
(a) $GF(p^n)$ p p a prime, $n > 0$	$GF(p^n) - \{0\}$	p^n	$p^n - 1$
(b) $\mathbb{Z}/(p^n)$ p an odd prime, $n \geqq 2$	$\{\ \bar{i} \mid 0 \leqq i \leqq p^n - 1,\ p \nmid i\}$	p^n	$(p-1)p^{n-1}$
(c) $GF(p)[X]/(X^2)$	$\{a + b\bar{X} \mid 0 \neq a \in GF(p), b \in GF(p)\}$ $= (GF(p) \setminus \{0\}) + GF(p)\bar{X}$	p^2	$(p-1)p$
(d) $\mathbb{Z}/(4)$	$\{\bar{1},\ \bar{3}\}$	4	2
(e) $GF(2)[x]/(X^3)$	$\{1,\ 1+\bar{X},\ 1+\bar{X}^2,\ 1+\bar{X}+\bar{X}^2\}$ $=< 1+\bar{X} >$	8	4
(f) $\mathbb{Z}[X]/(4, 2X, X^2-2)$	$\{\bar{1},\ \bar{3},\ \bar{1}+\bar{X},\ \bar{3}+\bar{X}\}$ $=< \bar{1}+\bar{X} >$	8	4
(g) $\begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$	8	2

Remark 19. *Let G_i be a finite cyclic group of order n_i (for $i = 1,\ 2$). Then $G_1 \times G_2$ is cyclic if and only if n_1 and n_2 are relatively prime.*

Except $GF(2^n)$ where n is a positive integer, all the other groups of units of rings in Theorem 14 have even orders. Hence, if a Gilmer ring is the direct sum of $k(\geqq 2)$ indecomposable rings, then at least $k-1$ of these component rings are $GF(2^{n_i})$ and $n_1,\ n_2, \cdots, n_{k-1}$ are relatively prime. For example, if $2^{n_1}-1,\ 2^{n_2}-1, \cdots, 2^{n_{k-1}}-1$ are distinct Mersenne primes, then $GF(5) \oplus GF(2^{n_1}) \oplus \cdots \oplus GF(2^{n_{k-1}})$ is a Gilmer ring.

Now we can prove the following:

Proposition 20. *R is a finite noncommutative indecomposable Gilmer ring if and only if (g) $\begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$, the ring of upper triangular matrices over $GF(2)$.*

Proof. Since R is a noncommutative indecomposable ring, $2^m R = 0$ for some $m > 0$. Since R is a Gilmer ring, R/J is also a Gilmor ring. Therefore R/J is a finite direct of fields. If R is a local ring, then R is commutative. Since R is not commutative, there is a nonzero idempotent $e \in R$ such that $e \neq 1$ and at least one of $eR(1-e)$ and $(1-e)Re$ is nonzero. Since $(eR(1-e))^2 = 0$ and $((1-e)Re)^2 = 0$, we see that $eR(1-e),\ (1-e)Re \subset J$. Hence $eR(1-e) = eJ(1-e)$ and $(1-e)Re = (1-e)Je$.

Considering R/J instead of R, we may assume that $J^2 = 0$. Since $J = eJe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$, we see that $1 + J = (1+eRe) \times (1+eR(1-e)) \times (1+(1-e)Re) \times (1+(1-e)R(1-e))$. Orders of $1+eRe,\ 1+eR(1-e),\ 1+(1-e)Re,\ 1+(1-e)R(1-e)$ are powers of 2 and one of $1+eR(1-e),\ 1+(1-e)Re$ is non-trivial. However $1+J$ is also cyclic, and hence only one of $1+eR(1-e),\ 1+(1-e)Re$ is nontrivial and other factors of $1+J$ is $\{1\}$. Hence $eJe = 0$, $(1-e)R(1-e) = 0$ and so $eRe \cong GF(2)$ and $(1-e)R(1-e) \cong GF(2)^k$ for some $k > 0$. Assume that $eR(1-e) \neq 0$. Then $eR(1-e)$ is a vctor space over $GF(2)$. Since $1+J$ is cyclic, $eR(1-e)$ is one dimensional over $GF(2)$. Since R is indecomposmble, $(1-e)R(1-e) \cong GF(2)$ and $R \cong \begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$.
If $eR(1-e) \neq 0$, then $R \cong \begin{pmatrix} GF(2) & 0 \\ GF(2) & GF(2) \end{pmatrix}$. However there is an isomorphism $\begin{pmatrix} GF(2) & 0 \\ GF(2) & GF(2) \end{pmatrix} \xrightarrow{\simeq} \begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$ by $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \longrightarrow \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$.

Some Characterizations of finite rings R by ratio $\dfrac{|R|}{|R^\times|}$

For a finite ring R, we define the number $\mu(R)$ by $\dfrac{|R|}{|R^\times|}$.

Theorem 21. *Let R be a finite ring with Jacobson radical J. Then the following are equivalent:*
(i) $R^{\times} = 1 + J$.
(ii) $\mu(R) \in \mathbb{Z}$.
(iii) $\mu(R) = 2^m$ *for some positive integer* m.
(iv) $R/J \cong GF(2)^{(m)}$ *for some positive integer* m.

A Boolean ring R is a ring for which $x^2 = x$ for all $x \in R$, that is, a ring that consists only of idempotent elements.

Since $1 + J \subset R^{\times}$, as a corollary of this theorem, we obtain Theorem 4.3 in [?].

Corollary 22. *Let R be a finite ring. Then $R^{\times} = 1$ if and only if R is a Boolean ring.*

Let $\mathbb{Z}_{(2)}$ denote the subring of $\mathbb{Q}$ consists of the form $\dfrac{a}{2^m}$ for some integer a and some non-negative integer m.

Theorem 23. *Let R be a finite ring. Then $\mu(R) = \dfrac{|R|}{|R^{\times}|} \in \mathbb{Z}_{(2)}$ if and only if R/J is isomorphic to a finite direct sum of copies of the following rings:*
(i) $GF(2)$,
(ii) $GF(p)$ *where p is a Fermat prime,*
(iii) $GF(3^2)$,
(iv) $M_2(GF(3))$.

Here remember that the Catalan's conjecture has been proven in 2002 by Preda Mihăilescu Mihăilescu's theorem in [?] is the following : The only solution in the natural numbers of $x^a - y^b = 1$ for $a, b > 1, x, y > 0$ is $x = 3,\ a = 2,\ y = 2,\ b = 3$. To prove the above theorem, I used Mihăilescu's theorem [?]. However, in our particular case, we expect there to be a more direct proof.

On finite rings whose groups of units are simple

Finally, for a finite ring R, we consider when $R^{\times}$ is a simple group. Although conditions for $R^{\times}$ to be a simple group was stated in Exercise 19.3 in [?], what is stated is not accurate. So we state the following.

Theorem 24. *Let R be a finite ring. Then $R^\times$ is a simple group if and only if R is isomorphic to one of the following rings:*
(i) $GF(3) \oplus GF(2)^{(k)}$ *for some non-negative integer k,*
(ii) $GF(2^n) \oplus GF(2)^{(k)}$ *for some non-negative integer k, where $p = 2^n - 1$ is a Fermat prime,*
(iii) $M_n(GF(2)) \oplus GF(2)^{(k)}$ *for some non-negative integer k and some $n \geqq 3$,*
(iv) $\mathbb{Z}/4\mathbb{Z} \oplus GF(2)^{(k)}$ *for some non-negative integer k,*
(v) $T \oplus GF(2)^{(k)}$ *for some non-negative integer k, where*
$$T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \subset M_2(GF(2)),$$
(vi) $\begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix} \oplus GF(2)^{(k)}$ *for some non-negative integer k.*

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ON THE CONSTRUCTION OF COHN'S UNIVERSAL LOCALIZATION

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Abstract

For an associative ring R we investigate a construction of Cohn's universal ring of fractions R_Σ, defined relative to a multiplicative set Σ of matrices. The construction of R_Σ avoids the Ore condition, which is necessary to construct a ring of fractions relative to a multiplicative set of elements of R. But a similar condition, which we call the "pseudo-Ore" condition, plays an important role in the construction of R_Σ. We show that this condition in fact determines the equivalence relation used in the construction of R_Σ, and provides information about left R_Σ-modules.

Throughout this paper, R will denote an associative ring with identity 1 that is not necessarily commutative, and X will denote a unital left R-module. The major new results are given in Theorems 11 and 12. The first concerns the equivalence relation used to define R_Σ, and the second characterizes the kernel of the canonical mapping from X to $R_\Sigma \otimes_R X$.

A subset $S \subset R$ is said to be a multiplicative set if it contains 1 and is closed under multiplication. Correspondingly, a set Σ of square matrices is said to be multiplicative if it contains all permutation matrices over R, is closed under multiplication (when defined), and if $C, D \in \Sigma$, then $\begin{bmatrix} C & A \\ 0 & D \end{bmatrix} \in \Sigma$ for any matrix A over R of the appropriate size. We denote the set of $n \times n$ matrices in Σ by Σ_n.

It is shown by Cohn in [3] that given a multiplicative set Σ of matrices over R, there exists a ring R_Σ and a ring homomorphism $\lambda : R \to R_\Sigma$ that is universal with respect to inverting the matrices in Σ. The ring R_Σ is constructed by adjoining enough elements to invert the given matrices, subject to the necessary relations. As pointed out in [3], this provides little information about $\ker(\lambda)$, and to address this, other constructions have been given by Malcolmson (see [6]) and Gerasimov (see [4]). Malcolmson's construction has been simplified by the present author in [2]. It is that construction which will be investigated here, as a way to provide further information about the kernels of the canonical mappings $\lambda : R \to R_\Sigma$ and $\mu_X : X \to R_\Sigma \otimes_R X$.

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The construction in [2] proceeds as follows. Cohn has shown that each element of R_Σ is an entry $e_i\lambda(C)^{-1}e_j^t$, in a matrix of the form $\lambda(C)^{-1}$, where $C \in \Sigma$, e_i, e_j are unit row vectors, and e_j^t denotes the transpose of e_j. Addition of triples requires us to model elements of the form $\lambda(a)\lambda(C)^{-1}\lambda(b)^t$, where $a, b \in R^n$, and $C \in \Sigma_n$. Since it is just as easy to construct a module of quotients, we consider ordered triples (a, C, x^t), where $a \in R^n$, $C \in \Sigma_n$, and $x \in X^n$, where X is any unital left R-module. Following Malcolmson's development in [6], we first define an addition on ordered triples.

Definition 1. *The sum of ordered triples* (a, C, x^t), (b, D, y^t), *with* $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$ *and* $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$ *is defined by*

$$(a, C, x^t) + (b, D, y^t) = \left([a \quad b], \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right).$$

The next step is to introduce the following equivalence relation (see Definition 2.1 of [2]) under which the equivalence classes of ordered triples form a commutative semigroup.

Definition 2. *Let* $a, b \in R^n$, $C_1, C_2 \in \Sigma_n$, *and* $x, y \in X^n$. *If there exist invertible* $n \times n$ *matrices* U_1, U_2 *over* R *such that* $a = bU_1$, $y^t = U_2 x^t$, *and* $C_2 U_1 = U_2 C_1$, *then we write*

$$(a, C_1, x^t) \equiv (b, C_2, y^t),$$

and we say that (a, C_1, x^t) *and* (b, C_2, y^t) *are* congruent *via* U_1, U_2.

Lemma 3 ([2])**.** *Under the congruence relation* $\equiv$, *addition of triples is commutative.*

Proof. If $C \in \Sigma_n$ and $D \in \Sigma_m$, then

$$(a, C, x^t) + (b, D, y^t) \equiv (b, D, y^t) + (a, C, x^t)$$

since

$$[a \quad b] = [b \quad a] \begin{bmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_n & 0 \end{bmatrix},$$

$$\begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_n & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_n & 0 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix},$$

and

$$\begin{bmatrix} y^t \\ x^t \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I}_m \\ \mathbf{I}_n & 0 \end{bmatrix} \begin{bmatrix} x^t \\ y^t \end{bmatrix},$$

where $\mathbf{I}_n$ and $\mathbf{I}_m$ are identity matrices of the appropriate sizes. ∎

We denote by $\Sigma^{-1}X_0$ the subsemigroup generated by all triples of the form $(0, C, x^t)$ or $(b, D, 0^t)$, for $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$ and all $n > 0$. Since addition is commutative, the elements of $\Sigma^{-1}X_0$ can be put in the form $(0, C, x^t) + (b, D, 0^t)$.

Definition 4. *Let $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$ and $b \in R^m$, $D \in \Sigma_m$, $x \in X^m$. If there exist $z_1, z_2 \in \Sigma^{-1}X_0$ such that $(a, C, x^t) + z_1 \equiv (b, D, y^t) + z_2$, then we write*

$$(a, C, x^t) \sim (b, D, y^t)\,.$$

The equivalence classes of ordered triples under the equivalence relation $\sim$ will be denoted by $[a : C : x^t]$, and $\Sigma^{-1}X$ will denote the set of all such equivalence classes.

Proposition 2.3 of [2] shows that $\sim$ defines a congruence on the semigroup of ordered triples, and that $\Sigma^{-1}X$ is an abelian group. With an appropriate multiplication, it is then shown in [2] that $\Sigma^{-1}R$ is a ring isomorphic to the universal localization R_Σ, and that $\Sigma^{-1}X$ is a left module over R_Σ that is naturally isomorphic to $R_\Sigma \otimes_R X$.

To construct a ring of left fractions using a multiplicative set $S \subset R$, we need to be able to replace any product $a_1c_1^{-1}$ with a product $c_2^{-1}a_2$, where $a_1, a_2 \in R$ and $c_1, c_2 \in S$ (see [5]). This leads to the left Ore condition: given $a_1 \in R$ and $c_1 \in S$, there exist $a_2 \in R$ and $c_2 \in S$ such that $c_2a_1 = a_2c_1$.

Addition in R_Σ does not require the left Ore condition. However, by a fundamental result (see Lemma 2.4 of [2]), if $a \in R^m$, $C_1 \in \Sigma_n$, $C_2 \in \Sigma_m$, $x \in X^n$, then

$$(aA_1, C_1, x^t) \sim (a, C_2, A_2x^t)$$

for any $m \times n$ matrices A_1, A_2 over R such that $C_2A_1 = A_2C_1$. This looks like the left Ore condition, since in R_Σ we are replacing $\lambda(A_1)\lambda(C_1)^{-1}$ by $\lambda(C_2)^{-1}\lambda(A_2)$. We will use this condition to define a new relation. Note the important point that the matrices C_1, C_2 may have different sizes.

Definition 5. *Let $a \in R^n$, $C_1 \in \Sigma_n$, $x \in X^n$ and $b \in R^m$, $C_2 \in \Sigma_m$, $y \in X^m$. Suppose that there exist $m \times n$ matrices A_1, A_2 over R and factorizations $a = bA_1$, $y^t = A_2x^t$. If $C_2A_1 = A_2C_1$, then we write $(a, C_1, x^t) \geq (b, C_2, y^t)$.*

In this case, we say that $(a, C_1, x^t) \geq (b, C_2, y^t)$ via A_1, A_2.

Lemma 6. *The relation $\geq$ is reflexive, transitive, and respects addition.*

Proof. Using the identity matrix, it follows immediately that $\geq$ is reflexive.

To show that the transitive law holds, let $(a_1, C_1, x_1^t) \geq (a_2, C_2, x_2^t)$ via A_1, A_2 and let $(a_2, C_2, x_2^t) \geq (a_3, C_3, x_3^t)$ via B_2, B_3. Then $a_1 = a_2A_1$, $x_2^t = A_2x_1^t$, and $C_2A_1 = A_2C_1$ in the first case, and $a_2 = a_3B_2$, $x_3^t = B_3x_2^t$, and $C_3B_2 = B_3C_2$ in the second case. Substituting yields $a_1 = a_2 A_1 = a_3(B_2 A_1)$, $x_3^t = B_3 x_2^t = (B_3 A_2) x_1^t$, and $C_3(B_2 A_1) = (C_3 B_2) A_1 = (B_3 C_2) A_1 = B_3(C_2 A_1) = B_3(A_2 C_1) = (B_3 A_2) C_1$. It follows that $(a_1, C_1, x_1^t) \geq (a_3, C_3, x_3^t)$ via $B_2 A_1, B_3 C_2$, showing that $\geq$ is a transitive relation.

To show that $\geq$ respects addition, suppose that $(a_1, C_1, x_1^t) \geq (a_2, C_2, x_2^t)$ via A_1, A_2. If (a_3, C_3, x_3^t) is any ordered triple, then

$$(a_1, C_1, x_1^t) + (a_3, C_3, x_3^t) \geq (a_2, C_2, x_2^t) + (a_3, C_3, x_3^t)$$

via the matrices $\begin{bmatrix} A_1 & 0 \\ 0 & \mathbf{I} \end{bmatrix}$ and $\begin{bmatrix} A_2 & 0 \\ 0 & \mathbf{I} \end{bmatrix}$. □

The next proposition is Lemma 2.4 of [2]. For the sake of completeness, we include the proof here.

Proposition 7 (Left pseudo-Ore condition). *Let $a \in R^n$, $C_1 \in \Sigma_n$, $x \in X^n$, $b \in R^m$, $C_2 \in \Sigma_m$, $y \in X^m$. If $(a, C_1, x^t) \geq (b, C_2, y^t)$, then $[a : C_1 : x^t] = [b : C_2 : y^t]$.*

Proof. We have

$$[a \quad aA_1] = [a \quad 0] \begin{bmatrix} \mathbf{I}_m & A_1 \\ 0^t & \mathbf{I}_n \end{bmatrix}, \quad \begin{bmatrix} \mathbf{I}_m & A_2 \\ 0 & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} 0 \\ x^t \end{bmatrix} = \begin{bmatrix} A_2 x^t \\ x^t \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & A_1 \\ 0 & \mathbf{I}_n \end{bmatrix} = \begin{bmatrix} \mathbf{I}_m & A_2 \\ 0 & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix},$$

so by definition we have

$$\left([a \quad aA_1], \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix}, \begin{bmatrix} 0^t \\ x^t \end{bmatrix}\right) \equiv \left([a \quad 0], \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix}, \begin{bmatrix} A_2 x^t \\ x^t \end{bmatrix}\right)$$

via $U_1 = \begin{bmatrix} \mathbf{I}_m & A_1 \\ 0 & \mathbf{I}_n \end{bmatrix}$ and $U_2 = \begin{bmatrix} \mathbf{I}_m & A_2 \\ 0 & \mathbf{I}_n \end{bmatrix}$. It follows from the definition of $\sim$ that $(a, C_2, 0^t)$ and $(0, C_1, x^t)$ act as neutral elements for addition, and therefore

$$\begin{aligned} (aA_1, C_1, x^t) &\sim (a, C_2, 0^t) + (aA_1, C_1, x^t) \\ &= \left([a \quad aA_1], \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix}, \begin{bmatrix} 0^t \\ x^t \end{bmatrix}\right) \\ &\equiv \left([a \quad 0], \begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix}, \begin{bmatrix} A_2 x^t \\ x^t \end{bmatrix}\right) \\ &= (a, C_2, A_2 x^t) + (0, C_1, x^t) \\ &\sim (a, C_2, A_2 x^t). \end{aligned}$$

This completes the proof, since, by definition, $\equiv$ implies $\sim$, and $\sim$ is transitive. □

Example 1. For any vectors b, x and matrices $C, D \in \Sigma$ we have $(0, C, x^t) \geq (b, D, 0^t)$ via the zero matrices of the appropriate size. On the other hand, if $b \neq 0$ we cannot have $(b, D, 0^t) \geq (0, C, x^t)$ since there does not exist a matrix A_1 with $b = 0A_1$, and, similarly, if $x \neq 0$, there does not exist a matrix A_2 with $x^t = A_2 0^t$. □

The above example shows that we need to introduce a right pseudo-Ore condition. Note that because the matrices U_1, U_2 in the definition of the congruence relation $\equiv$ are invertible, the relation $\equiv$ is in fact symmetric.

Definition 8. *Let $a \in R^n$, $C_1 \in \Sigma_n$, $x \in X^n$ and $b \in R^m$, $C_2 \in \Sigma_m$, $y \in X^m$. Suppose that there exist $n \times m$ matrices A_1, A_2 over R and factorizations $x^t = A_1 y^t$, $b = aA_2$. If $C_1 A_2 = A_1 C_2$, then we write $(a, C_1, x^t) \leq (b, C_2, y^t)$.*

In this case, we say that $(a, C_1, x^t) \leq (b, C_2, y^t)$ via A_1, A_2.

The relation $\leq$ is reflexive, transitive, and respects addition (the proofs are dual to those in Lemma 6). Part (b) of the next proposition establishes that the right pseudo-Ore condition holds in $\Sigma^{-1}X$.

Proposition 9. *Let $a \in R^n$, $C_1 \in \Sigma_n$, $x \in X^n$, $b \in R^m$, $C_2 \in \Sigma_m$, $y \in X^m$.*

(a) *We have $(a, C_1, x^t) \geq (b, C_2, y^t)$ if and only if $(b, C_2, y^t) \leq (a, C_1, x^t)$.*

(b) *If $(a, C_1, x^t) \leq (b, C_2, y^t)$, then $[a : C_1 : x^t] = [b : C_2 : y^t]$.*

Proof. (a) By a careful application of the definitions, $(a, C_1, x^t) \geq (b, C_2, y^t)$ via A_1, A_2 if and only if $a = bA_1$, $y^t = A_2 x^t$, and $C_2 A_1 = A_2 C_1$ and $(b, C_2, y^t) \leq (a, C_1, x^t)$ via B_1, B_2 if and only if $y^t = B_1 x^t$, $a = bB_1$, and $C_2 B_2 = B_1 C_1$. Thus $(a, C_1, x^t) \geq (b, C_2, y^t)$ via A_1, A_2 if and only if $(b, C_2, y^t) \leq (a, C_1, x^t)$ via A_2, A_1.

(b) This follows immediately from part (a) and Proposition 7. □

Lemma 10. *Let $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$, $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$. Then*

(a) $(a, C, x^t) \geq (a, C, x^t) + (b, D, 0^t)$ *and* $(a, C, x^t) + (0, D, y^t) \geq (a, C, x^t)$;

(b) $(a, C, x^t) + (b, D, 0^t) \leq (a, C, x^t)$ *and* $(a, C, x^t) \leq (a, C, x^t) + (0, D, y^t)$.

Proof. (a) Since $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} C$, by definition we have

$$\begin{aligned} (a, C, x^t) = \left([a \;\; b] \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}, C, x^t \right) &\geq \left([a \;\; b], \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} x^t \right) \\ &= (a, C, x^t) + (b, D, 0^t) \,. \end{aligned}$$

Since $C\,[\mathbf{I} \quad 0] = [\mathbf{I} \quad 0] \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$, by definition we have

$$\begin{aligned} (a, C, x^t) + (0, D, y^t) &= \left(a\,[\mathbf{I} \quad 0], \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right) \\ &\geq \left(a, C, [\mathbf{I} \quad 0] \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right) = (a, C, x^t)\,. \end{aligned}$$

(b) This follows immediately from part (a) and Proposition 9 (a). □

The following theorem shows that the left and right "pseudo-Ore" conditions determine the equivalence relation $\sim$ used in the construction of $\Sigma^{-1}X$.

Theorem 11. *Let n, m be positive integers, let $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$, and let $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$. The following conditions are equivalent:*

(1) $(a, C, x^t) \sim (b, D, y^t)$;

(2) *there exist $u \in R^k$, $E, F \in \Sigma_k$, $z \in X^k$, for some positive integer k, such that $(a, C, x^t) + (0, E, z^t) \geq (b, D, y^t) + (u, F, 0^t)$;*

(3) *there exist triples (a_1, C_1, x_1^t) and (b_1, D_1, y_1^t) such that $(a, C, x^t) \leq (a_1, C_1, x_1^t)$, $(a_1, C_1, x_1^t) \geq (b_1, D_1, y_1^t)$, and $(b_1, D_1, y_1^t) \leq (b, D, y^t)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $(a, C, x^t) \sim (b, D, y^t)$. Then by definition there exist triples $(0, E, z^t)$, $(u_2, F_2, 0^t)$, $(0, E_2, z_2^t)$, $(u, F, 0^t)$ such that

$$(a, C, x^t) + (0, E, z^t) + (u_2, F_2, 0^t) \equiv (b, D, y^t) + (0, E_2, z_2^t) + (u, F, 0^t)\,.$$

Since the relation $\equiv$ respects addition, if $E \in \Sigma_j$ and $F \in \Sigma_k$ with $j < k$, then we can add $k - j$ copies of $(0, 1, 0)$ to both $(0, E, z^t)$ and $(0, E_2, z_2^t)$ while maintaining the given identity. A similar argument can be given if $j > k$, so without loss of generality we can assume that $j = k$.

It follows from Lemma 10 (a) that

$$(a, C, x^t) + (0, E, z^t) \geq (a, C, x^t) + (0, E, z^t) + (u_2, F_2, 0^t)$$

and that

$$(b, D, y^t) + (0, E_2, z_2^t) + (u, F, 0^t) \geq (b, D, y^t) + (u, F, 0^t)\,.$$

Since $\geq$ is transitive by Lemma 6 and $\equiv$ implies $\geq$, we have

$$(a, C, x^t) + (0, E, z^t) \geq (b, D, y^t) + (u, F, 0^t)\,.$$

(2) $\Rightarrow$ (3): Given the triples $(0, E, z^t)$ and $(u, F, 0^t)$ in condition (2), let

$$(a_1, C_1, x_1^t) = (a, C, x^t) + (0, E, z^t)$$

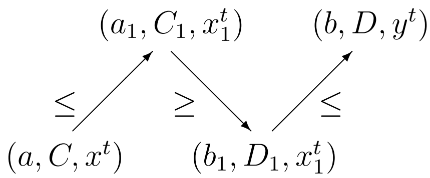
and

$$(b_1, D_1, y_1^t) = (b, D, y^t) + (u, F, 0^t)\,.$$

Then $(a, C, x^t) \le (a_1, C_1, x_1^t)$ by Lemma 10 (b), $(a_1, C_1, x_1^t) \ge (b_1, D_1, y_1^t)$ by hypothesis, and $(b_1, D_1, y_1^t) \le (b, D, y^t)$ by Lemma 10 (a).

(3) $\Rightarrow$ (1): This follows immediately from Propositions 9 and 7. □

In the following diagram, we denote $\le$ by showing the first triple below the second. Then $(a, C, x^t) \sim (b, D, y^t)$ if and only if there there exist triples (a_1, C_1, x_1^t) and (b_1, D_1, y_1^t) such that the following relationship holds.



Corollary 7.11.9 of [3] states if $r \in R$, then $r \in \ker(\lambda)$ if and only if for some $C, D \in \Sigma$ there is a relation of the form

$$\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ C & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ D & B_{22} \end{bmatrix}.$$

We call this Gerasimov's criterion, since it has been developed in [4]. Using Theorem 11, we can extend it to modules. We let μ_X denote the canonical mapping $\mu_X : X \to \Sigma^{-1}X$ defined by setting $\mu_X(x) = [1 : 1 : x]$, for all $x \in X$.

Theorem 12. *The following conditions are equivalent for the inversive set Σ and $x \in X$:*

(1) *$x \in \ker(\mu_X)$, for the canonical mapping $\mu_X : X \to \Sigma^{-1}X$;*

(2) *there exist $a \in R^n$, $z \in X^n$, and $C, D \in \Sigma_n$, for some $n > 0$, such that*

$$(1, 1, 0) + (0, D, z^t) \ge (1, 1, x) + (a, C, 0^t)\,;$$

(3) *there exist relations of the form*

$$\begin{bmatrix} x \\ 0^t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ C & A \end{bmatrix} \begin{bmatrix} y^t \\ z^t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} \\ C & A \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for vectors a_{11}, a_{12} over R, y, z over X, and matrices A, B, C, D such that $C, D \in \Sigma$.

Proof. (1) $\Rightarrow$ (2): Since $x \in \ker(\mu_X)$ if and only if $(1,1,0) \sim (1,1,x)$, this is a direct application of Theorem 11.

(2) $\Rightarrow$ (3): Suppose that $(1,1,0) + (0,D,z^t) \geq (1,1,x) + (a,C,0^t)$ via A_1, A_2. Writing A_1 and A_2 in block form, there exist $a_{11}, b_{11} \in R$, $a_{12}, a_{21}, b_{12}, b_{21} \in R^n$, and $A, B \in M_n(R)$ such that

$$[1\ \ 0] = [1\ \ a] \begin{bmatrix} a_{11} & a_{12} \\ a_{21}^t & A \end{bmatrix} \qquad \begin{bmatrix} b_{11} & b_{12} \\ b_{21}^t & B \end{bmatrix} \begin{bmatrix} 0 \\ z^t \end{bmatrix} = \begin{bmatrix} x \\ 0^t \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21}^t & A \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21}^t & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}.$$

This gives us the following equations:

$$a_{11} + a \cdot a_{21}^t = 1 \qquad a_{12} + aA = 0 \qquad b_{12} \cdot z^t = x \qquad Bz^t = 0^t$$

$$a_{11} = b_{11} \qquad a_{12} = b_{12}D \qquad Ca_{21}^t = b_{21}^t \qquad CA = BD.$$

Substituting $a_{12} = b_{12}D$ in the equation $a_{12} + aA = 0$, we only need to use the following equations in order to obtain the desired result:

$$aA + b_{12}D = 0 \qquad CA - BD = 0 \qquad b_{12} \cdot z^t = x \qquad Bz^t = 0^t.$$

These equations show that

$$\begin{bmatrix} a & b_{12} \\ C & -B \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & b_{12} \\ C & -B \end{bmatrix} \begin{bmatrix} 0 \\ z^t \end{bmatrix} = \begin{bmatrix} x \\ 0^t \end{bmatrix}.$$

(3) $\Rightarrow$ (1): Suppose that there are relations of the form

$$\begin{bmatrix} x \\ 0^t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ C & A \end{bmatrix} \begin{bmatrix} y^t \\ z^t \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} \\ C & A \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $C \in \Sigma_n$. $D \in \Sigma_m$, A, B are $n \times m$ matrices over R, $a_{11} \in R^n$, $a_{12} \in R^m$, $y \in X^n$, and $z \in X^m$. Then $CB = -AD$, and so it follows from the left pseudo-Ore condition that

$$(-a_{11}B, D, z^t) \sim (-a_{11}, C, -Az^t).$$

Since $-a_{11}B = a_{12}D$, we have

$$(-a_{11}B, D, z^t) = (a_{12}D, \mathbf{I}_m D, z^t) \sim (a_{12}, \mathbf{I}_m, z^t),$$

by an easy application of the left pseudo-Ore condition. Similarly, we have $-Az^t = Cy^t$, and so

$$(a_{11}, C, -Az^t) = (a_{11}, C\mathbf{I}_n, Cy^t) \sim (a_{11}, \mathbf{I}_n, y^t)\,.$$

Therefore the sum

$$(a_{11}, \mathbf{I}_n, y^t) + (a_{12}, \mathbf{I}_m, z^t) \sim (a_{11}, C, -Az^t) + (-a_{11}, C, -Az^t) \sim (a_{11} - a_{11}, C, -Az^t)$$

must belong to $\Sigma^{-1}X_0$.

It follows easily from the left pseudo-Ore condition that $(a, I_n, y^t) \geq (1, 1, a \cdot y^t)$ and $(a, C, y^t) + (a, C, z^t) \geq (a, C, (y + z)^t)$, for any $a \in R^n$, $C \in \Sigma_n$, and $y, z \in X^n$. We conclude that

$$(1, 1, x) = (1, 1, a_{11}\cdot y^t + a_{12}\cdot z^t) \sim (1, 1, a_{11}\cdot y^t) + (1, 1, a_{12}\cdot z^t) \sim (a_{11}, \mathbf{I}_n, y^t) + (a_{12}, \mathbf{I}_m, z^t)$$

belongs to $\Sigma^{-1}X_0$, and therefore $x \in \ker(\mu_X)$. □

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10

On certain algebraic structures associated with Lie (super)algebras

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Introduction

This note is to deal with a cetain survey based on our papers mainly and to give several examples of triple systems, by using this triple's consept we exhibit a construction of simple Lie algebras, furthermore to describe a history of Jordan rivers in nonassociative algebras from the author's viewpoint. On the other hand, this work with respect to Jordan and Lie structures is in close contact with a symmetric (super)space equipped with commplex structure, since the tangent space of the symmetric (super)space is a δ-Lie triple system ($\delta = \pm 1$).

From mathematical history's viewpoint, the concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$. Nonassociative algebras are rich in algebraic structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry (for example, [11], [46], [59], [61], [64]). Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([5], [10]-[13], [15], [22], [26], [28], [29], [33], [34], [35], [45], [54], [55], [60], [65]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [6]-[8], [20], [23], [24], [40]-[45], [50]-[53], [56]-[58]) by using the standard embedding method ([22], [48], [49], [57], [63]). In particular, the generalized Jordan triple system of second order, or $(-1,1)$-Freudenthal Kantor triple system (for short $(-1,1)$-FKTS), is a useful concept ([13]-[21], [41]-[44], [47], [62]) for the constructions of simple Lie algebras, while the $(-1,-1)$-FKTS plays the same role ([6], [22], [25], [27], [30], [32], [33]) for the construction of Lie superalgebras, while the δ-Jordan Lie triple systems act similarly for that of Jordan superalgebras ([23], [24], [56]). Specially, we have constructed a model of basic Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ ([22], [25], [27]).

As a final comment of this introduction, we provide well-known results due to O. Loos as follows; if A is a unital commutative Jordan algebra with a unital element e, that is, satisfying $(xy)x^2 = x(y(x^2))$, and $xy = yx$, then the triple product given by

$$\{xyz\} = (xy)z + x(yz) - y(xz)$$

defines a Jordan triple system with $\{xey\} = xy$, i.e., it satisfies the two relations $\{xy\{abc\}\} = \{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$ (this relation is often called a

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fundamental identity), and $\{xyz\} = \{zyx\}$ (this relation is called a commutative identity, since $xy = \{xey\} = \{yex\} = yx$) and next the new triple product $[xyz]$ given by

$$[xyz] = \{xyz\} - \{yxz\}$$

defines a Lie triple system.

Briefly summarizing this article, we will generalize these results and exhibit examples of Lie (super)algebras associated with generalized Jordan triple systems. Toward to its applications, in particular, we will give a construction of symmmetric (super)spaces with an almost complex structure (i.e., eqipped with Nijenhuis operator). And we will exhibit an idea of bisymmetric spases associated with our constructions.

Roughly describing, we have an illustration for our concept ;

Algebraic structures $\Longleftrightarrow$ Geometric structures.

For examples, it seems that there are certain algebraic structures associated with symmetric, R-symmetric, homogeneous spaces, totally geodesic manifold, and symmetric domains, etc.

1 Definitions and Results

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order) ([41]-[45]) .

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1}$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \tag{2}$$

where $L(a, b)c := (abc)$ and $K(a, b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the following condition

$$(abc) = (cba), \ i.e., \ K(a, c)b = 0. \tag{3}$$

The JTS is a special case in the GJTS of 2nd order since $K(x, y) \equiv 0$.

We next can generalize the concept of GJTS of 2nd order as follows (see [13], [14], [18], [22], [28], [36] [63] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{4}$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0, \tag{5}$$

where

$$L(a, b)c := (abc), \quad K(a, b)c := (acb) - \delta(bca), \tag{6}$$

is called an $(\varepsilon, \delta)-Freudenthal-Kantor$ *triple system* (for short (ε, δ)-FKTS). An (ε, δ)-FKTS is said to be *unitary* if $Id \in \{K(a, b)\}_{span}$.

A triple system satisfying only the identity (4) is called a *generalized* FKTS (for short GFKTS), while the identity (5) is called the *second order condition* (this condition needs to construct of 5-graded Lie (super)algebras).
Remark. From the relation Eq. (6), we note that

$$K(b,a) = -\delta K(a,b). \tag{7}$$

A triple system is called a (α, β, γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha < x, y > z + \beta < y, z > x + \gamma < z, x > y,$$

where $< x, y >$ is a bilinear form such that $< x, y >= \kappa < y, x >$, $\kappa = \pm 1$, $\alpha, \beta, \gamma \in \Phi$.

From now on we will mainly consider this type of triple system.

An (ε, δ)-FKTS is said to be *balanced* if there is a bilinear form $< x, y >\in \Phi^*$ such that $K(x, y) =< x, y > Id$, that is, $dim\ \{K(x, y)\}_{span} = 1$ holds.
Remark. We note that a balanced triple system (i.e., it fulfills $K(x, y) =< x, y > Id$) is unitary, since $Id \in \{K(x, y)\}_{span}$.

Triple products are denoted by (xyz), $\{xyz\}$, $[xyz]$ and $< xyz >$ upon their suitability.
Remark. We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$-FKTS. Thus we can construct the corresponding Lie algebras by means of the standard embedding method ([6], [13]-[19], [21], [22], [25], [27], [43]).
For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc], a, b, c \in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{aligned} [abc] &= -\delta[bac], \\ [abc] + [bca] + [cab] &= 0, \\ [ab[xyz]] &= [[abx]yz] + [x[aby]z] + [xy[abz]], \end{aligned} \tag{8}$$

where $a, b, x, y, z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS*, by ([14]). Note that the set $L(V, V)$ of all left multiplications $L(x, y)$ of V is a Lie subalgebra of $Der\ V$, where we denote by $L(x, y)z = [xyz]$.
Proposition 1.1. ([13]-[16], [22]) *Let $(U(\varepsilon, \delta), < xyz >)$ be an (ε, δ)-FKTS. If J is an endomorphism of $U(\varepsilon, \delta)$ such that $J < xyz >=< JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon, \delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] :=< xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > . \tag{9}$$

Remark. Note that for the case of $\varepsilon = -1, \delta = 1$ and $K(x, y) = 0$, we have a special case in Prop.1.1, that is, it implies that $J = Id$, $\{xyz\}$ is the JTS and $[xyz] = \{xyz\} - \{yxz\}$ is the LTS described in Introduction.
Corollary. ([13]) *Let $U(\varepsilon, \delta)$ be an (ε, δ)-FKTS. Then the vector space $T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}. \tag{10}$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(U(\varepsilon,\delta)) = D(T(\varepsilon,\delta),T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated with $T(\varepsilon,\delta)$ where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$;

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)) := \left\{ \begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix} \right\}_{span},$$

$$T(\varepsilon,\delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x,y \in U(\varepsilon,\delta) \right\}_{span}.$$

We use the following notation:

$$\mathbf{k} := \{K(x,y) \in End\ U(\varepsilon,\delta) | x,y \in U(\varepsilon,\delta)\}\ and$$

$$\{EFG\} := EFG + GFE,\ \forall E,F,G \in \mathbf{k}.$$

Then, we may make the structure of a JTS $\mathbf{k}$ with respect to the triple product $\{EFG\} \in \mathbf{k}$, hence $[EFG] = \{EFG\} - \{FEG\}$ has a structure of LTS ([20]).

We next introduce an analogue of Nijenhuis tentor in differential geometry defined by

$$N(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] + J^2[X,Y], \forall X,Y \in T(\varepsilon,\delta)$$

and $J = \begin{pmatrix} 0 & \varepsilon \\ -\delta & 0 \end{pmatrix}$, that is $J^2 = -\varepsilon\delta Id$, hence if $J^2 = -Id$, then this (the case of $\varepsilon\delta = 1$) has a structure of allmost complex.

Proposition 1.2. *Let U be a (ε,δ)-FKTS, $T(\varepsilon,\delta)$ be the δ-LTS and $L(U)$ be the standard embedding Lie (super)algebra associated with U. Then the following are equivalent:*

(i) $N(X,Y) = 0, \forall X,Y \in T(\varepsilon,\delta)$,

(ii) $\varepsilon\delta L(y,x) - \varepsilon L(x,y) = K(x,y),\ \forall x,y \in U(\varepsilon,\delta)$.

This $J \in End\ T(\varepsilon,\delta)$ may generalize on $\tilde{J} \in End\ L(U)$ defined by

$$\tilde{J} := JD(X,Y)J^{-1} \oplus JZ, \quad \forall X,Y,Z \in T(\varepsilon,\delta).$$

Then we note that $\tilde{J}$ has an interesting property, for example, an automorphism of $L(U)$ associated with U.

Proposition 1.3. *For a (ε,δ)-FKTS U and $L(U)$ as in above Proposition, assuming $\varepsilon = \delta$ and $K(x,y) = L(y,x) - \varepsilon L(x,y)$, then the elements $f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in sl(2)$ (i.e., $[f,g] = h,\ [f,h] = -2f,\ [g,h] = 2g$) are derivations of $L(U)$.*

Remark. We note that $L(U) = L(U(\varepsilon,\delta)) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra such that $U(\varepsilon,\delta) \oplus U(\varepsilon,\delta) = L_{-1} \oplus L_1 = T(\varepsilon,\delta)$ (δ-LTS), $L_{-2} = \mathbf{k}$ (JTS) and $D(T(\varepsilon,\delta),T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$ (the derivation of $T(\varepsilon,\delta)$) equipped with $[L_i,L_j] \subseteq L_{i+j}$ and $L_{-1} \oplus L_1 = L(U)/L_{-2} \oplus L_0 \oplus L_2$. In Introduction, we had used the notation $g = g_{-1} \oplus g_0 \oplus g_1$ instead of $L_{-1} \oplus L_0 \oplus L_1$. This Lie (super)algebra construction is one of reasons to study nonassociative algebras and triple systems without using root systems (for a Lie superalgebra, refer to ([9], [12], [60])). Also this construction can be represented by the concept of a normal triality algebra (see [34], [35]).

This section is a survey of our papers with respect to the triple systems and the construction of Lie (super)algebras mainly, if it need, the readers would like to see the earlier our references therein.

2 Examples of (ε, δ)-JTS

We will consider here examples of the special case defined by bilinear forms $< x, y >$, that is, an (ε, δ)-JTS of (α, β, γ) triple systems equipped with $K(x, y) \equiv 0$. Moreover, we give two examples (Prop. 2.2 and Prop.2.3) without the cases of (ε, δ)-JTS.

Example 2.1. Let V be a vector space with a symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z+ < y, z > x- < z, x > y$$

defines on V a $(-1, 1)$-JTS.
Note that $(-1, 1)$-JTS is same as the JTS.

Example 2.2. Let V be a vector space with an anti-symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z+ < y, z > x- < z, x > y$$

defines on V a $(1, -1)$-JTS.

Example 2.3. Let V be a vector space with a symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z- < y, z > x$$

defines on V a $(-1, -1)$-JTS.

Example 2.4. Let V be a vector space with an anti-symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z- < y, z > x$$

defines on V a $(1, 1)$-JTS.

Example 2.5. Let V be a set of alternative matrix $Asym(n, \Phi) = \{x|^t x = -x\}$, where ${}^t x$ denote the transpose matrix of x. Then

$$< xyz >= x^t yz - \varepsilon z^t yx, \quad where \ \forall \ x, y, z \in V$$

defines on V a $(\varepsilon, -\varepsilon)$ JTS, that is, the case of $\varepsilon = -1 \Rightarrow$ JTS.

Remark. Let V be the set of $p \times q$ matrix $Mat(p, q; \Phi)$. Then this vector space V is a JTS with respect to the product $\{xyz\} = x^t yz + z^t yx, \ \forall x, y, z \in V$.

Proposition 2.1. *Let $(U, < xyz >)$ be an (ε, δ)-JTS. Then the triple system is a δ-LTS with respect to the new product*

$$[xyz] =< xyz > -\delta < yxz > . \tag{11}$$

In the next section 3 subsection we study the case of an (ε, δ)-FKTS, but we give first two examples which are not (ε, δ)-JTS as it follows.

Proposition 2.2. *Let $(U, < xyz >)$ be a triple system with $< xyz >=< y, z > x$ and $< x, y >= -\varepsilon < y, x >$. Then this triple system is an (ε, δ)-FKTS.*

Proposition 2.3. ([16], [18]) *Let U be a balanced $(1, 1)$-FKTS satisfying $<< xxx >, x >\equiv 0$ (identically) and $< x, y >$ is nondegenerate. Then U has a triple product defined by*

$$< xyz >= \frac{1}{2}(< y, x > z+ < y, z > x+ < x, z > y). \tag{12}$$

Note that the balanced $(1,1)$-FKTS induced from an exceptional Jordan algebra is closely related to the 56 dimensional meta symplectic geometry due to H. Freudenthal ([13], [15], [16], [18] and the earlier references therein). Also the correspondence of a quaternionic symmetric space and the balanced (1,1) FKTS has been studied in ([5]). On the other hand, for $(-1,-1)$ -FKTS, see ([6] and[7], [30], [31]).

3 Examples of Lie (super)algebras associated with (ε,δ) Freudenthal-Kantor triple systems

We will exhibit the examples of some triple systems and Lie (super)algebras associated with their triple systems. Unless otherwise stated, all Lie (super)algebras considered here are complex and finite dimensional.
Example a). $C(n+1)$ type is of dimension $\dim C(n+1) = 2n^2 + 5n + 1$.

Let U be the set of matrices $M(1, 2n; \Phi)$. Then, by Example 2.2, it follows that the triple product

$$L(x,y)z =< xyz >:=< x,y > z+ < y,z > x- < z,x > y$$

such that the bilinear form fulfills $< x,y >= - < y,x >$, is a $(1,-1)$-JTS, since $K(x,y) \equiv 0$ (identically). Furthermore, the standard embedding Lie superalgebra is 3-graded and of $C(n+1)$ type. For the extended Dynkin diagram, we obtain

$$L_{-1} \oplus L_0 \oplus L_1 := \left\{ \begin{pmatrix} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{pmatrix} \middle| \varepsilon = 1 = -\delta \right\}_{span} \oplus \left\{ \begin{pmatrix} e \\ f \end{pmatrix} \right\}_{span} \cong$$

$$\begin{array}{l} \otimes\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \qquad\quad \alpha_n \quad \alpha_{n+1} \\ \ \ \| \ \ > \ \ \circ - \circ - - - - - - \circ <= \circ \\ \otimes\ \alpha_0 \end{array}$$

$$= C(n+1)\ type\ (\alpha_1\ \otimes\ deleted).$$

Also, we obtain

$$L_0 := \left\{ \begin{pmatrix} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{pmatrix} \middle| \varepsilon = 1 = -\delta \right\}_{span} \cong$$

$$\begin{array}{l} \alpha_2 \quad \alpha_3 \qquad\quad \alpha_n \quad \alpha_{n+1} \\ \circ - \circ - - - - - - \circ <= \circ \end{array}$$

$$= C_n \oplus \Phi Id\ (\alpha_1\ \otimes\ and\ \alpha_0\ \otimes\ deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $C(n+1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \otimes$.
Example b). $B(n,1)$ and $D(n,1)$ type are of dimension $\dim B(n,1) = 2n^2 + 5n + 5$ and $\dim D(n,1) = 2n^2 + 3n + 3$, respectively.

Let U be the set of matrices $M(1, l; \Phi)$. Then, by straihtfoward calculations, it follows that the triple product

$$L(x,y)z =< xyz >:= \frac{1}{2}(< x,y > z- < y,z > x+ < z,x > y)$$

such that the bilinear form fulfills $< x, y > = < y, x >$ is a $(-1, -1)$-FKTS. Furthermore, the standard embedding Lie superalgebra is 5-graded and of $B(n, 1)$ type if $l = 2n + 1$, or of $D(n, 1)$ type if $l = 2n$. For the extended Dynkin diagram, we obtain from the results of § 1 the following.

For the case of $B(n, 1)$ type we have

$$L_{-2} \oplus L_0 \oplus L_2 := D(T(-1, -1), T(-1, -1)) =$$

$$\left\{ \begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix} \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\begin{array}{c} \alpha_0 \quad \alpha_1 \quad \alpha_2 \qquad\qquad\quad \alpha_n \quad \alpha_{n+1} \\ \circ => \otimes - \circ - - - - - - \circ => \circ \end{array}$$

$$= A_1 \oplus B_n \ type \ (\alpha_1 \ \otimes \ deleted).$$

Also, we obtain

$$L_0 := \left\{ \begin{pmatrix} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{pmatrix} \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\begin{array}{c} \alpha_2 \quad \alpha_3 \qquad\quad \alpha_n \quad \alpha_{n+1} \\ \circ - \circ - - - - - - \circ => \circ \end{array}$$

$$= B_n \oplus \Phi Id \ (\alpha_1 \ \otimes \ and \ \alpha_0 \ \circ \ deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $B(n, 1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \circ$.

Similarly, for the case of $D(n, 1)$ type we have $L_{-2} \oplus L_0 \oplus L_2 \cong A_1 \oplus D_n$, $L_0 \cong D_n \oplus \Phi Id$. We note that this triple system is balanced and with a complex structure of Nijenhuis tensor zero, since $K(x, y) = < x, y > Id = L(x, y) + L(y, x)$ (c.f. [36]).

Remark. The examples a),b) are simple triple systems, since the bilinear forms $< x, y >$ are nondegenerate. Indeed, if $I \neq 0$ is an ideal of U then, by straightforward calculations, from the fact that $< I, U > \ U \subseteq I$ and $< \ , \ >$ is nondegenerate, we have $I = U$. Hence U is simple.

Remark. We note that the case of balanced is discussed in ([18], [28]). On the other hand, for the construction of simple exceptional Lie algebras G_2, F_4, E_6, E_7, E_8, refer to ([16], [18], [21]). Also, for the construction of simple Lie superalgebras $G(3)$, $F(4)$, $D(2, 1, \alpha), P(n), Q(n), H(n)$, $S(n)$ and $W(n)$, refer to ([22], [25], [27], [31]). Of course, these construction are created from the concept of triple systems without using systems of roots. Thus, moreover, these examples imply that our methods may apply the symmetric superspace (the case of $\delta = -1$) as well as the structures (see, [5], [46]) of the symmetric spaces (the case of $\delta = 1$), however we will not go into the details.

In the rest of this section, we will consider the constructions of simple B_3-type Lie algebra associated with several triple systems (the case of $\varepsilon = -1$ and $\delta = 1$), more easily. That is, we will give several examples; (c) the case of a JTS (i.e., $(-1, 1)$-FKTS with $K(x, y) \equiv 0$), (d) the case of a GJTS of 2nd order (i.e., $(-1, 1)$-FKTS with $dim\{K(x, y)\}_{span} = 1$), (e) the case of a GJTS of 2nd order (i.e, $(-1, 1)$-FKTS with $dim\{K(x, y)\}_{span} = 3$).

Example c). We study the case of $g_{-1} = U = Mat(1, 5; \Phi)$. Hereafter in this section, as a reason of traditional notation, we often would like to denote by g_i instead of $L_i, (i = 0, \pm 1, \pm 2)$ and by $\{xyz\}$ instead of $< xyz >$.

In this case, g_{-1} is a JTS with respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy, \ \forall \ x, y, z \in g_{-1}$$

where ${}^t x$ denotes the transpose matrix of x.

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_3-type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and a LTS $T(U) = U \oplus U = g_{-1} \oplus g_1$. Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong B_2 \oplus \Phi H, \ where\ H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Here in view of the relations $[S(x, y), L(a, b)] = L(S(x, y)a, b)+ L(a, S(x, y)b)$, and $[A(x, y), L(a, b)] = L(A(x, y)a, b) - L(a, A(x, y)b)$ for all $L(a, b) \in End\ U$, when $\varepsilon = -1, \delta = 1$, we use the following notations;

$$Der\ U := \{L(x, y) - L(y, x)\}_{span},$$

$$Anti - Der\ U := \{L(x, y) + L(y, x)\}_{span},$$

$$g_0 = \left\{ \begin{pmatrix} L(x, y) & 0 \\ 0 & -L(y, x) \end{pmatrix} \right\}_{span} = \left\{ S(x, y) + A(x, y) \right\}_{span}$$

where $S(x, y) := L(x, y) - L(y, x) \in Der\ U,\ A(x, y) := L(x, y) + L(y, x) \in Anti - Der\ U$, this case is $\varepsilon = -1\ and\ \delta = 1$.

Example d). We study the case of $g_{-1} = U = Mat(2, 3; \Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1, 1)$-FKTS) with $dim\ \{K(x, y)\}_{span} = 1$ with respect to the product

$$\{xyz\} = x^t yz + z^t yx - z^t xy, \ \forall \ x, y, z \in g_{-1}.$$

By straightforward calculations, it can be shown that the standard embedding Lie algebra $L(U) = g$ is a 5-graded B_3-type Lie algebra with $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and $dim\ g_{-2} = dim\ g_2 = dim\ \{K(x, y)\}_{span} = 1$. Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong A_1 \oplus A_1 \oplus \Phi H, \ where\ H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Furthermore, we obtain a LTS $T(U)$ of $dim\ T(U) = dim\ (g_{-1} \oplus g_1) = 12$,

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_1 \oplus A_1 \oplus A_1 \cong Der\ T(U).$$

Also, in this case, we note that $T(U) = L(U)/Der\ T(U) = g/(g_{-2} \oplus g_0 \oplus g_2)(= g_{-1} \oplus g_1)$ is the tangent space of a quaternion symmetric space of dimension 12, since $T(U)$ is a Lie triple system associated with g_{-1}.

Example e). Third, we study the case of $g_{-1} = U = Mat(1, 3; \Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1, 1)$-FKTS) with respect to the product

$$\{xyz\} = x^t yz + z^t yx - y^t xz, K(x, y)z = \{xzy\} - \{yzx\}, \ \forall \ x, y, z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 5-graded B_3-type Lie algebra with $g = g_{-2} \oplus \cdots \oplus g_2$ and $dim\ g_{-2} = dim\ g_2 = 3$. Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong A_2 \oplus \Phi H,\ g_{-2} = \{K(x,y)\}_{span} = Alt(3,3;\Phi).$$

Furthermore, we obtain a LTS $T(U)$ of $dim\ T(U) = dim\ (g_{-1} \oplus g_1) = 6$,

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_3 \cong Der\ T(U).$$

This case $g_{-2} = \{K(x,y)\}_{span} = \mathbf{k}$ has the structure of a JTS (cf. section 2).
Remark. We remark that the cases (a)and (b) (resp. (c),(d),(e)) are $\delta = -1$ (resp. $\delta = 1$).
Remark. For the root system $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$and the highest root $-\rho = \{\alpha_1 + 2\alpha_2 + 2\alpha_3\}$ of the simple Lie algebra B_3, the case of (c) means that $g_{-1} = \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+2\alpha_3, \alpha_1+2\alpha_2+2\alpha_3\}$ and $g_{-2} = \{0\}$, the case of (d) means that $g_{-1} = \{\alpha_2, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_2+2\alpha_3\}$ and $g_{-2} = \{-\rho\}$, the case of (e) means that $g_{-1} = \{\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3\}$ and $g_{-2} = \{\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$.

4 Mathematical physics Remarks

In this section, we give several references of mathematical physics in our works. We note that there are applications toward the Yang-Baxter equations associated with triple systems ([26], [39], [57]) and also toward the field theory associated with Hermitian triple systems ([37], [38]). For other mathematical physics, it seems that the books ([28], [33]) are useful.

5 History from a certain personal viewpoint

For a mathematical history, in particular for Jordan rivers, we describe belows: This brief history (with respect to nonassociative algebras) is a story from author's personal aspect (judgement). Triple systems (ternary algebras) have first been appeared from Prof. N. Jacobson and continued by Profs. O. Loos, K. Meyberg and E. Neher of students of Prof. M. Koecher in Germany, also certain triple systems associated with the geometry of 56 dimensional due to Prof. Freudenthal have been studied by Prof. J. Faulkner (resp. K. Meyberg) of the student of Prof. N. Jacobson (resp. Prof. M. Koecher).

On the other hand, there is a history;

H. Freudenthal (Netherlands) $--$ >K. Yamaguti (Japan) or I. L. Kantor (Russian and Sweden, he was born in Belarus) $-- >$ Author (N. Kamiya) $-- >$ D. Mondoc (but these arrows are no students), however, Dr. Mondoc is only a student of Prof. Kantor in Sweden.
Profs. O. Loos and E. Neher in the student of Prof. M. Koecher in Germany are working in Jordan triple systems and Jordan pairs. Profs. Kantor, Yamaguti, S. Okubo and author(N. Kamiya) are studying in their generalizations, for example, refer to N. Kamiya and S. Okubo *"Representation of* (α, β, γ) *triple*

systems,", Linear and Multilinear Algebras, **58** no.5-6 (2010) 617-643. This history is a story whithout using concept of root systems and Cartan matrix in Lie algebras, in particular, is a study for triple systems.

Note that there are a lot of mathematician in nonassociative algebras (for Lie algebras), but a little groups in triple systems or Jordan algebras. For example, Profs. E. Zelmanov, K. McCrimmon, B. Allison, V. Kac, I. Shestakov, H. Petersson, M. Racine, H. Asano, I. Satake, M. C. Myung, A. Elduque, C. Martinez, S. Gonžalez, S. Okubo and author, may be, only a few. Furthermore in addition, the book *"A Taste of Jordan Algebras"* (Springer, 2003) written by Prof. K. McCrimmon of a student in N. Jacobson is described about a history of the Jordan river. It here emphasize that this historical survey of certain Jordan algebras until the end of the 20th century and the beginning of 21th century is my (author) aspect (viewpoint). In addition to above river, for a certain example, for our imaginative illustrations with respect to a generalization of numbers;

$$(\sharp)\ \mathbf{R} \to \mathbf{C} \to \mathbf{H} \to \mathbf{O}(octonion) \to \mathbf{H_3}(\mathbf{O})(Jordan\ algebra\ of\ 27\ dim) \to$$

$$\mathbf{M}(\mathbf{H_3}(\mathbf{O}))(metasymplectic\ geometry\ of\ 56\ dim) \to$$

$$\mathbf{T}(\mathbf{H_3}(\mathbf{O}))(symmetric\ space\ of\ 112\ dim) \to$$

$$\mathbf{E_8}(exceptional\ \ simple\ Lie\ algebra\ of\ 248\ dim).$$

On the other hand, there is other river also,

$$(\sharp\sharp)\ \mathbf{O} \to \mathbf{C} \otimes \mathbf{O},\ \mathbf{H} \otimes \mathbf{O}\ and\ \mathbf{O} \otimes \mathbf{O}\ (Freudenthal's\ magic\ square) \to$$

$$\mathbf{T}(\mathbf{O} \otimes \mathbf{O})(symmetric\ space\ of 128\ dim) \to \mathbf{E_8}.$$

For another way, there is a river of Prof. Tits (called Tits's construction) as follows.

($\sharp\sharp\sharp$) The case $A = \mathfrak{A}_0 \otimes \mathfrak{J}_0$; (with $dim\ A = 7 \times 26,\ dim\ Der\ (A) = 66$), where the base field Φ is an algebraically closed field of characteristic 0.

$L(A) = Der(A) \oplus A \cong E_8,\ Der(A) = Der\mathfrak{A} \oplus Der\mathfrak{J} \cong G_2 \oplus F_4 =< D(X,Y) >_{span}$. Here $\mathfrak{A}_0$ denote $\{x \in \mathbf{O} | trace\ x = 0\}$ and $\mathfrak{J}_0 = \{x \in H_3(\mathbf{O}) | Trace\ x = 0\}$. For the product of A, $X \circ Y = (a * b) \otimes (x * y)$ and with respect to the Lie product of $L(A)$, $[X,Y] = D(X,Y) + X \circ Y$, then the vector space $(A,\ \circ)$ has an algebraic structure of satisfying $D(X \circ Y, Z) + D(Y \circ Z, X) + D(Z \circ X, Y) = 0$, where $X, Y, Z \in A$ ([19] and see the earlier references therin).

If we set $\mathfrak{J} = H_3(\mathbf{O}) \to H_3(\mathfrak{A})\ (= B)$, then we have the following table;

	$dimB = 1$	$dimB = 6$	$dimB = 9$	$dimB = 15$	$dimB = 27$
$dim\mathfrak{A} = 1$	0	A_1	A_2	C_3	F_4
$dim\mathfrak{A} = 2$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$dim\mathfrak{A} = 4$	A_1	C_3	A_5	D_6	E_7
$dim\mathfrak{A} = 8$	G_2	F_4	E_6	E_7	E_8

Here note that $L(A)/(G_2 \oplus F_4)$ is a reductive homogeneous space with 182 dimension.

It seems that there are several researchers group's tradition for these study and furthermore, for a nonassociative world of 21th century, Spanish, Portuguese and middle Europe scholars groups will glow with respect to the study (may be, Prof. Elduque's group mainly).

For algebraic structures of nonassociative subject (AMS classification 17) related with geometry, about 20th century, roughly speaking, we may describe as follows, for example (in my opinion);

Jordan algebras researchers (E. Artin origin),

Lie algebras researchers (N. Jacobson origin).

In summarizing about Jordan algebras or triple systems, we have the following diagrams (a generalization of complex and quaternionic numbers):

octonion, pseudo octonion algebras and triple systems $\Longrightarrow$

Jordan algebras +Lie (super)algebras +symmetric composition algebras

$\Longrightarrow$**mathematical algebras** (author's new phrase)

In final comments (although they had described in the introduction), also we emphasize that nonassociative algebras are rich in algebraic structures, and they provide important common ground for various branches of mathematics, not only pure algebra and mathematical physics (for example, Pierce decompositions, Yang-Baxter equations and quark theory), but also analysis (Jordan C^* algebras or JB^* triple), topology (racks or quandles), and geometries (generalized symmetric spaces, convex cones or bounded symmetric domaines, in particular). Hence, in future aspect, it seems that the triple systems (or ternary product) without using unit elements are useful concept for several subjects of sciences as well as the situation of symmetric spaces.

6 Geometric structures

6.1 A generalized curvature and torsion tensors

Let $L = L(U(\varepsilon,\delta)) = L(W,W) \oplus W$ be the Lie algebra defined from a δ- LTS as in the section one, that is, the δ-LTS $W = T(\varepsilon,\delta) = L_{-1} \oplus L_1$ is induced from $L_{-1} = U(\varepsilon,\delta)$ (as L_{-1} has the structure of a (ε,δ)-FKTS), where $\varepsilon,\delta = \pm 1$.

We now introduce a generalization of covariant derivative $\bigtriangledown$ in differential geometry as follows; $\bigtriangledown : L \to \mathrm{End}\ L$ defined by

$\bigtriangledown_X Y = [X,Y] = -\delta[Y,X]$,

$\bigtriangledown_X [Y,Z] = [YZX] = -\delta[ZYX]$,

$\bigtriangledown_{[X,Y]} Z = -[XYZ] = \delta[YXZ]$,

$\bigtriangledown_{[X,Y]}[V,Z] = [[V,Z],[X,Y]] = -\delta[[X,Y],[V,Z]]$, for any $X,Y,Z,V \in W$.

Furtheremore, a generalized curvature tensor defined by

$$C_\delta(X,Y) = \bigtriangledown_X \bigtriangledown_Y - \delta \bigtriangledown_Y \bigtriangledown_X - \bigtriangledown_{[X,Y]} \tag{13}$$

is identically zero, i.e., $C_\delta(X,Y) = 0$ in L, for any $X,Y \in W$. Indeed, we demonstrate the proof below.

First we calculate

$$C_\delta(X,Y)Z = (\bigtriangledown_X \bigtriangledown_Y - \delta \bigtriangledown_Y \bigtriangledown_X)Z - \bigtriangledown_{[X,Y]} Z$$

$$= \bigtriangledown_X [Y,Z] - \delta \bigtriangledown_Y [X,Z] + [XYZ]$$

11

$$= [YZX] - \delta[XZY] + [XYZ]$$
$$= [YZX] + [ZXY] + [XYZ] = 0.$$

Second, it follow

$$C_\delta(X,Y)[V,Z] = (\bigtriangledown_X \bigtriangledown_Y - \delta \bigtriangledown_Y \bigtriangledown_X)[V,Z] - \bigtriangledown_{[X,Y]}[V,Z]$$
$$= [X,[VZY]] - \delta[Y,[VZX]] + \delta[[X,Y],[V,Z]]$$
$$= [X, L(V,Z)Y] - \delta[Y, L(V,Z)X] - L(V,Z)[X,Y] = 0$$

(by $[Y, L(V,Z)X] = -\delta[L(V,Z)X, Y]$ and $[[X,Y],[V,Z]] = -\delta[[V,Z],[X,Y]]$) for any $X,Y,Z,V \in T(\varepsilon,\delta)$.

However a generalized torsion tensor defined by

$$S_\delta(X,Y) = \bigtriangledown_X Y - \delta \bigtriangledown_Y X - [X,Y] \tag{14}$$

is not zero, since it gives $S_\delta(X,Y) = [X,Y] - \delta[Y,X] - [X,Y] = [X,Y]$.

Note that the case of $\delta = 1$ is appeared in ([36]).

In final comments of this section, for δ-LTS $W = T(\varepsilon,\delta)$, we recall the Nijenhuis operator in the section one;

$$N(X,Y) = [JX,JY] + J^2[X,Y] - J[JX,Y] - J[X,JY],$$

where J is an almost complex structure on W, this concept (the case of $\delta = 1$) is appeared in ([36]), hence we may consider a generalization with respect to the super symmetric space (the case of $\delta = -1$).

If we set $J = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, or $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then we have $J^2 = -Id,\ or\ J^2 = Id$ respectively, and it seems that there is a twisted or a straight (para complex) property in the sence of W. Bertram.

6.2 Magic square table of exceptional simple Lie algebras

Following ([34],[35]), note that we can construct the exceptional simple Lie algebras E_6, E_7, E_8, F_4 *and* G_2 associated with pre-structurable or normal triality algebras A, that is, the construction of 5-graded Lie algebras $L(A) = g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and $g_{-1} = A$. Of course, as the construction in section one, we have certain structure of triple systems (ternary algebras) with rspect to the algebra A.

We now denote that the base field Φ is an algebraically closed field of characteristic 0, and a Cayley algebra by **O**.

I) $A = \mathbf{O} \otimes \mathbf{O}$ (tensor product case, $dim\ A = 64$, $dim\ g_{-2} = dim\ g_2 = 14$). For subalgebras of A, if we use the notation of $A = A_1 \otimes A_2,\ dim\ A_1, dim\ A_2$, then the Lie algebras obtained from their subalgebras are following:

$L(A) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \cong E_8,\ g_0 \cong D_7 \oplus gl(1),\ g_{-2} \oplus g_0 \oplus g_2 \cong D_8, A = g_{-1}$

$dimA_2 \backslash dimA_1$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

For this case's E_8, considering the Extended Dynkin diagram, we have

$g_{-1} \oplus g_1 \cong L(A)/(g_{-2} \oplus g_0 \oplus g_2) = E_8/D_8$ with dim $(g_{-1} \oplus g_1)$=128;

⊚ − ∘ − ∘ − ∘ − ∘ − ∘ − ∘−[∘], [∘] omitted $\cong$ D_8, and ⊚ is highest root.
|
∘

II) $A = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ (balanced case, $dim\ A = 56$, $dim\ g_{-2} = dim\ g_2 = 1$), where $\mathbf{a},\ \mathbf{b} \in H_3(\mathbf{O})$ (exceptional Jordan algebra with 27 dimension) and $\alpha,\ \beta \in F$. The Lie algebra constructed from this algebra A is the following.
$L(A) \cong E_8,\ g_{-2} \oplus g_0 \oplus g_2 \cong E_7 \oplus A_1,\ g_0 \cong E_7 \oplus gl(1),\ A = g_{-1}$.
To change the notation $H_3(\mathbf{O}) \to H_3(\mathfrak{A})(= B)$, here $\mathfrak{A}$ is Hurwitz algebras over F. $\forall \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in \begin{pmatrix} F & B \\ B & F \end{pmatrix} = A$, with respect to the $dimB$, Lie algebras $L(A)$ obtained from B are the following.

$dimB$	1	6	9	15	27
$dimA$	4	14	20	32	56
$dimL(A)$	14	52	78	133	248
$L(A)$	G_2	F_4	E_6	E_7	E_8

For this case's E_8, considering the Extended Dynkin diagram, we have
$g_{-1} \oplus g_1 \cong L(A)/(g_{-2} \oplus g_0 \oplus g_2) = E_8/(A_1 \oplus E_7)$ with dim $g_{-1} \oplus g_1$=112;

⊚− [∘]− ∘ − ∘ − ∘ − ∘ − ∘ −∘, [∘] omitted $\cong$ $A_1 \oplus E_7$.
|
∘

Remark. This construction of type II with $dim\ A = 56$ has been first studied by H. Freudenthal (called a metasymmetric geometry equipped with notations of $P \times Q$ and $\{P, Q\}$). And this concept is characterized by a triple system (or a ternary algebra) called a generalized Zorn's vector matrix ([13]-[16],[18],[35] the references of therein).

6.3 Bisymmetric spaces associated with exceptional simple Lie algebras

Following the books due to O. Loos or W. Bertram with respect to symmetric spaces, it is known to have associated to a symmetric space $M = G/H$ a Lie triple system T (as the tangent space of the symmetric space is a Lie triple system).

We consider a concept of bisymmetric space $(B_\alpha, B_\beta, B_\gamma, B_\delta)$ in Lie triple subsystems pair defined as follows:

(I) *dim* B_δ/*dim* B_γ = *dim* B_γ/*dim* B_β = *dim* B_β/*dim* $B_\alpha = 2$, *and*

$$B_\alpha < B_\beta < B_\gamma < B_\delta$$

as Lie triple subsystem's series of the Lie triple system $g_{-1} \oplus g_1$ of 5-graded Lie algebra $g = L(A) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ associated the normal triality algebra $A = g_{-1}$ and $Der\ (g_{-1} \oplus\ g_1) \cong g_{-2} \oplus g_0 \oplus g_2$.

From § 6.2 (I) type, we obtain bisymmetric space's series of type (I). It is said to be a type (I) bisymmetric space.

$$(\heartsuit)\quad F_4/B_4 < E_6/(D_5 \oplus gl(1)) < E_7/(D_6 \oplus A_1) < E_8/D_8$$

and the dimensin of bisymmetric spaces of type (I);

$$16, \quad 32, \quad 64, \quad 128 \qquad respectively.$$

With respect to Extended Dynkin diagrams (symmetric spaces) of E_7, E_6, F_4;

$\circ - \boxed{\circ} - \circ - \circ - \circ - \circ - \circledcirc$, $\boxed{\circ}$ omitted $\cong D_6 \oplus A_1$, and $\circledcirc$ is highest root.

$\qquad |$

$\qquad \circ$

$\boxed{\circ} - \circ - \circ - \circ - \boxed{\circ}$, $\boxed{\circ}$ omitted $\cong D_5 \oplus gl(1)$, and $\circledcirc$ is highest root.

$\qquad |$

$\qquad \circ$

$\qquad |$

$\qquad \circledcirc$

$\circledcirc - \circ - \circ => \circ - \boxed{\circ}$, $\boxed{\circ}$ omitted $\cong B_4$, and $\circledcirc$ is highest root.

From § 6.2 (II) type, we may define the same concept to type (I) as follows.

$$(II) \quad (dim\ B_\delta + 16)/dim\ B_\gamma = (dim\ B_\gamma + 16)/dim\ B_\beta =$$

$$(dim\ B_\beta + 16)/dim\ B_\alpha = 2,\ and\ B_\alpha < B_\beta < B_\gamma < B_\delta$$

It is said to be a type(II) bisymmetric space.

$$(\heartsuit\heartsuit) \quad F_4/(C_3 \oplus A_1) < E_6/(A_5 \oplus A_1) < E_7/(D_6 \oplus A_1) < E_8/(E_7 \oplus A_1)$$

and the dimension of bisymmetric spaces of type (II);

$$28, \quad 40, \quad 64, \quad 112 \qquad respectively.$$

With respect to Extended Dynkin diagrams (symmetric spaces) type (II) of E_7, E_6, F_4;

$\circ - \circ - \circ - \circ - \circ - \boxed{\circ} - \circledcirc$, $\boxed{\circ}$ omitted $\cong D_6 \oplus A_1$, and $\circledcirc$ is highest root.

$\qquad |$

$\qquad \circ$

$\circ - \circ - \circ - \circ - \circ$, $\boxed{\circ}$ omitted $\cong A_5 \oplus A_1$, and $\circledcirc$ is highest root.

$\qquad |$

$\qquad \boxed{\circ}$

$\qquad |$

$\qquad \circledcirc$

$\circledcirc - \boxed{\circ} - \circ => \circ - \circ$, $\boxed{\circ}$ omitted $\cong C_3 \oplus A_1$. and $\circledcirc$ is highest root.

Here $A_1, A_5, B_4, C_3, D_5, D_6$ mean classical simple Lie algebras.

Remark. For type (II), we consider with $L(A)/g_0$ vector spaces series;

$$F_4/(C_3 \oplus gl(1)) < E_6/(A_5 \oplus gl(1)) < E_7/(D_6 \oplus gl(1)) < E_8/(E_7 \oplus gl(1))$$

$(dim\ B_\delta + 18)/dim\ B_\gamma = (dim\ B_\gamma + 18)/dim\ B_\beta = (dim\ B_\beta + 18)/dim\ B_\alpha = 2,$ $and\ B_\alpha < B_\beta < B_\gamma < B_\delta.$

This bivector spaces series have dimensions 30, 42, 66, 114 respectively, (may be, it seems that there is a certain algebraic structure, perhaps, to be said a bireductive homogeneous space).

The content of this section is a new idea and the details will be described a forthcoming paper in future.

Concluding Remark. One of fundamntal our philosophy is to study the construction of 5-graded Lie (super)algebras $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$, satisfying $[g_i, g_j] \subseteq g_{i+j}$without using roots systems and Cartan matrix.

In the end of this paper, for the references of this subject (nonassociative algebras) recently, we note that it is useful to refer a Springer publisher book (Math. and Statistics series, vol. 427) with respect to the Proceeding of conferences of Nonassociative algebras and its applications in Coimbra University (2022, Portgal).

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These references are mainly papers for our study fields (as a survey article in these fields, so we have a lot of references).

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REVERSE LEGENDRE POLYNOMIALS

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ABSTRACT. For n a nonnegative integer, let $\mathscr{P}_n$ be the vector space of polynomials of degree at most n, equipped with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)dx$. Let $\mathscr{B} = \{x^n, x^{n-1}, \ldots, 1\}$, an ordered basis of $\mathscr{P}$. Let $\{\overleftarrow{P}_n^n(x), \overleftarrow{P}_{n-1}^n(x), \ldots, \overleftarrow{P}_0^n(x)\}$ be the orthogonal basis of $\mathscr{P}_n$ obtained by applying the Gram-Schmidt procedure to $\mathscr{B}$, with the resulting polynomials normalized by the condition that $\overleftarrow{P}_k^n(1) = 1$. We call these polynomials reverse Legendre polynomials. In this paper we give explicit formulas for the reverse Legendre polynomials $\{\overleftarrow{P}_k^n(x)\}$ and determine some of their properties.

We begin with the vector space $\mathscr{P}$ of all polynomials, equipped with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)dx$. As is well known, the Legendre polynomials $P_0(x), P_1(x), \ldots$ are the polynomials obtained by applying the Gram-Schmidt procedure to the ordered basis $\mathscr{B} = \{1, x, x^2, \ldots\}$ of $\mathscr{P}$, except that the resulting polynomials are normalized by the condition that $P_n(1) = 1$ for every n. (We cite [1, Chapter 22] for all facts about and properties of Legendre polynomials.) We think of these polynomials as being obtained by applying the Gram-Schmidt procedure "from the bottom up", and it seemed to us a natural question to ask what happens if we apply this procedure "from the top down". Of course, $\mathscr{B}$ has no "top". But instead, for any nonnegative integer n, we may let $\mathscr{P}_n$ be the vector space of polynomials of degree at most n, equipped with the same inner product, and apply the Gram-Schmidt procedure to the ordered basis $\mathscr{B} = \{x^n, x^{n-1}, \ldots, 1\}$ of $\mathscr{P}_n$ to obtain polynomials $\{\overleftarrow{P}_n^n(x), \overleftarrow{P}_{n-1}^n(x), \ldots, \overleftarrow{P}_0^n(x)\}$ with the analogous normalization $\overleftarrow{P}_k^n(1) = 1$ for every n, k. Since these polynomials are obtained by reversing the order of the basis elements, we call these reverse Legendre polynomials. It is our objective here to explicity determine the polynomials $\{\overleftarrow{P}_k^n(x)\}$ and to describe some of their properties.

We proceed as follows. In section 1 we give the explicit formulas for $\{\overleftarrow{P}_k^n(x)\}$ and prove some of the properties of these polynomials. In sections 2 and 3 we prove that our formulas are correct. These formulas are admittedly not very enlightening, so in section 4 we give a table of values of $\{\overleftarrow{P}_k^n(x)\}$, which enable the reader to much better see what is going on. Indeed, the results of this paper were obtained by doing extensive calculations to obtain these values, then looking intently at them to see a pattern, and finally proving that the pattern we see is correct. Finally, in response to a question from the referee, in section 5 we give an application of reverse Legendre polynomials to quadrature.

It is not surprising that the formulas for the coefficients of $\{\overleftarrow{P}_k^n(x)\}$ involve the coefficients of the Legendre polynomials. But, as we shall see, it is only the odd Legendre polynomials that are involved.

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1. Generalities and statement of results

We begin by establishing a notational convention: The symbol Σ' will denote a summation over odd values of the index.

First we recall the formulas for the odd Legendre polynomials. (See [1, 22.3.8], which gives the formulas for all the Legendre polynomials, odd or even.)

Theorem 1.1. *For a nonnegative integer n, the Legendre polynomial $P_{2n+1}(x)$ is the polynomial of degree $2n+1$*

$$P_{2n+1}(x) = \sum_{i=1}^{2n+1}{}' a_{2n+1,i}x^i$$

where

$$a_{2n+1,i} = (1/2^{2n+1})(-1)^{(2n+1-i)/2}\binom{2n+1+i}{2n+1}\binom{2n+1}{\frac{2n+1-i}{2}}$$

It is convenient to have an alternative expression for the coefficients.

Lemma 1.2.

$$a_{2n+1,i} = (1/2^{2n+1})(-1)^{(2n+1-i)/2}\frac{(2n+1+i)!}{\frac{2n+1+i}{2}!\frac{2n+1-i}{2}!i!}$$

Proof. Routine computation. □

Now we establish some properties of the reverse Legendre polynomials.

Theorem 1.3. *Let m, n and k be nonnegative integers with $k \leq n$.*
(a) The reverse Legendre polynomial $\overleftarrow{P}_k^n(x)$ is a polynomial of degree at most n whose low-order term is a nonzero multiple of x^k.
(b) $\overleftarrow{P}_k^n(x)$ is an even polynomial if k is even and an odd polynomial if k is odd.
(c) $\overleftarrow{P}_k^{2m+k+1}(x) = \overleftarrow{P}_k^{2m+k}(x)$.
(d) $\overleftarrow{P}_k^{2m+k}(x)$ is a polynomial of degree $2m+k$, with a zero of order k at $x=0$ and $2m$ simple zeroes at nonzero values of x, all of them real numbers symmetrically located around the origin and lying in the open interval $(-1,1)$.
(e) $\overleftarrow{P}_k^n(x)$ is a polynomial of degree n if $n-k$ is even and of degree $n-1$ if $n-k$ is odd.
(f) The reverse Legendre polynomial $\overleftarrow{P}_k^n(x)$ is uniquely determined by condition (a) above and by the conditions that $\langle \overleftarrow{P}_j^n(x), \overleftarrow{P}_k^n(x)\rangle = 0$ for $j > k$ and that $\overleftarrow{P}_k^n(1) = 1$.

Proof. (a) Let us begin by recalling the Gram-Schmidt procedure in general. Let V be a vector space of finite or countably infinite dimension, and let $\mathscr{B} = \{v_1, v_2, \ldots\}$ be an ordered basis of V. The Gram-Schmidt procedure recursively produces an orthogonal (rep. orthonormal) basis $\mathscr{C} = \{w_1, w_2, \ldots\}$ (resp. $\tilde{\mathscr{C}} = \{\tilde{w}_1, \tilde{w}_2, \ldots\}$) of V as follows: $w_1 = v_1$ and, assuming that w_j is defined for $1 \leq j \leq k-1$, $w_k = v_k - \sum_{j=1}^{k-1}(\langle v_k, w_j\rangle)/(\langle w_j, w_j\rangle)w_j$ (and $\tilde{w}_k = (1/\langle w_k, w_k\rangle)w_k$). In particular, $w_k = v_k +$ a linear combination of $v_1, \ldots, v_{k-1}$ and consequently $\mathrm{Span}(\{w_1, \ldots, w_k\}) = \mathrm{Span}(\{v_1, \ldots, v_k\})$ for every $k \geq 1$.

Applying that procedure to the ordered basis $\mathscr{B} = \{x^n, x^{n-1}, \ldots, 1\}$ of $\mathscr{P}_n$ we see that $\overleftarrow{P}_k^n(x)$ is as claimed.

(b) This follows immediately from the fact that if $f(x)$ is any even function and $g(x)$ is any odd function, $\langle f(x), g(x)\rangle = 0$.

(c) $\overleftarrow{P}_k^{2m+k+1}(x)$ is obtained from $\{x^{2m+k+1},\ldots,x^k\}$ via the Gram-Schmidt procedure, and $\overleftarrow{P}_k^{2m}(x)$ is obtained from $\{x^{2m+k},\ldots,x^k\}$ via the Gram-Schmidt procedure. Comparing the two applications of this procedure, we see the results are exactly the same, by the proof of (b).

(d) By (a) and (b), we have that $\overleftarrow{P}_k^{2m+k}(x) = x^k f(x)$ for some even polynomial $f(x)$ of degree at most $2m$ with nonzero constant term. Thus the zeroes of $f(x)$ all occur at nonzero values of x, symmetrically located with respect to the origin. Let $f(x)$ have t zeroes of odd order $r_1,\ldots,r_t$ in the interval $(-1,1)$. We have that $t \le 2m$, so if we show that $t = 2m$ we will have established that these are all the zeroes of $f(x)$, that they are all simple, and that $\overleftarrow{P}_k^{2m+k}(x)$ has degree $2m+k$. We argue by contradiction. Suppose $t < 2m$. Since t is even, $t \le 2m-2$. Let $g(x) = (x-r_1)\cdots(x-r_t)$ and $h(x) = x^{k+2}g(x)$. Then on the one hand

$$\langle \overleftarrow{P}_k^{2m+k}(x), h(x)\rangle = \int_{-1}^{1} (x^k f(x))(x^{k+2}g(x))dx = \int_{-1}^{1} x^{2k+2} f(x)g(x)dx \neq 0,$$

as $x^{2k+2}f(x)g(x)$ has constant sign in $[-1,1]$ and is not identically zero. But on the other hand, $h(x)$ is a polynomial of degree at most $(k+2)+(2m-2) = 2m+k$ with low-order term of degree $k+2$, so $\overleftarrow{P}_k^{2m+k}(x)$ is orthogonal to $h(x)$, i.e., $\langle \overleftarrow{P}_k^{2m+k}(x), h(x)\rangle = 0$; contradiction.

(e) Suppose $n-k$ is even. Set $n-k = 2m$. Then, by (d), $\overleftarrow{P}_k^n(x)$ has degree $2m+k = n$. The case $n-k$ odd then follows immediately from (c).

(f) The polynomial $\overleftarrow{P}_k^n(x)$ is a polynomial in V, the vector space spanned by $\{x^n,\ldots,x^k\}$, that is orthogonal to W, the subspace of V spanned by $\{x^n,\ldots,x^{k+1}\}$. The orthogonal complement U of W in V is 1-dimensional, so $U = \{cf(x)\}$ where $f(x)$ is any nonzero polynomial in U and c is an arbitrary constant. Then if $g(1) \neq 0$ for some, and hence every, nonzero element $g(x)$ of U, $g(x)$ will be specified by the value of $g(1)$. But that is the case by (d). □

There are some cases in which we can readily determine $\overleftarrow{P}_k^n(x)$.

Theorem 1.4.
(a) $\overleftarrow{P}_n^n(x) = x^n$ *for all* $n \ge 0$.
(b) $\overleftarrow{P}_{n-1}^n(x) = x^{n-1}$ *for all* $n \ge 1$.
(c) $\overleftarrow{P}_{n-2}^n(x) = \frac{2n-1}{2}x^n - \frac{2n-3}{2}x^{n-2}$ *for all* $n \ge 2$.
(d) $\overleftarrow{P}_0^{2m+1}(x) = \overleftarrow{P}_0^{2m}(x) = P_{2m+1}(x)/x$ *for all* $m \ge 0$.

Proof. (a) is immediate, (b) immediately follows from $\langle x^n, x^{n-1}\rangle = 0$, and (c) is a routine computation with the Gram-Schmidt procedure.

As for (d), let $n = 2m$ or $2m+1$. We first note that $P_{2m+1}(x)$ is an odd polynomial of degree $2m+1$ with nonzero x term, so the quotient $P_{2m+1}(x)/x$ is an even polynomial of degree $2m \le n$ with a nonzero constant term. We show that $P_{2m+1}(x)/x$ satisfies the conditions of Theorem 1.3(f) and hence that $\overleftarrow{P}_0^{2m}(x) = P_{2m+1}(x)/x$. By Theorem 1.3(a), we have that, for any j between 1 and n, $\overleftarrow{P}_j^n(x)$ is a polynomial of degree at most n that is divisible by x^j, so in particular, for any such j, $\overleftarrow{P}_j^n(x)/x$ is a polynomial of degree at most

$n-1 \leq 2m$. But then

$$\langle \overleftarrow{P}_j^n(x), P_{2m+1}(x)/x\rangle = \int_{-1}^{1} \overleftarrow{P}_j^n(x)(P_{2m+1}(x)/x)dx$$
$$= \int_{-1}^{1} (\overleftarrow{P}_j^n(x)/x)P_{2m+1}(x)dx = \langle \overleftarrow{P}_j^n(x)/x, P_{2m+1}(x)\rangle = 0,$$

as $P_{2m+1}(x)$ is orthogonal to every polynomial of degree at most $2m$.

Also, the value of the polynomial $P_{2m+1}(x)/x$ at $x=1$ is $P_{2m+1}(1)/1 = 1/1 = 1$. □

We now explicitly determine $\overleftarrow{P}_k^n(x)$ in general.

Theorem 1.5. *Let m and k be nonnegative integers. Then*

$$\overleftarrow{P}_k^{2m+k+1}(x) = \overleftarrow{P}_k^{2m+k}(x) = \sum_{i=1}^{2m+1}{}' \, w_{m,k,i}a_{2m+1,i}x^{k+i-1}$$

where

$$w_{m,k,i} = \prod_{s=0}^{k-1} \frac{i+2s+2m+2}{i+2s+2}$$

Equivalently, let n be a nonnegative integer and let k be a nonnegative integer with $k \leq n$. Then

$$\overleftarrow{P}_k^n(x) = \sum_{i=1}^{n-k+1}{}' \, w_{(n-k)/2,k,i}a_{n-k+1,i}x^{k+i-1} \quad \text{for } n-k \text{ even,}$$
$$\overleftarrow{P}_k^n(x) = \sum_{i=1}^{n-k}{}' \, w_{(n-k-1)/2,k,i}a_{n-k,i}x^{k+i-1} \quad \text{for } n-k \text{ odd.}$$

Proof. In light of Theorem 1.3(f), it suffices to show that $\{\overleftarrow{P}_k^n(x)\}_{k=0,\dots,n}$, as given by these formulas, are pairwise mutually orthogonal and that $\overleftarrow{P}_k^n(1) = 1$ for $k = 0,\dots,n$. We show these properties in Sections 2 and 3 below. □

It is convenient to have the following alternate expressions for $w_{m,k,i}$. Here for an odd positive integer ℓ, we set $\ell!! = (1)(3)\cdots(\ell)$, the product of the odd integers from 1 to ℓ.

Lemma 1.6.

$$w_{m,k,i} = \frac{i!!(i+2k+2m)!!}{(i+2k)!!(i+2m)!!}$$
$$= \frac{\binom{i+2k+2m}{2k}\binom{(i-1)/2+k}{k}}{\binom{i+2k}{2k}\binom{(i-1)/2+k+m}{k}}$$

Proof. Routine computation. □

We also have an alternate expression for the coefficients of the reverse Legendre polynomials.

Lemma 1.7.

$$w_{m,k,i}a_{2m+1,i} = (1/2^{2m+1})(-1)^{(2m+1-i)/2}\left(2\frac{\binom{i+2k+2m}{2m}\binom{2m}{m}\binom{m}{(i-1)/2}}{\binom{(i-1)/2+k+m}{m}}\right)$$

Proof. Routine computation. □

From Theorem 1.1 and Lemma 1.6 we see that $2^{2m+1}w_{m,k,i}a_{2m+1,i}$ is a rational number with odd denominator. On the basis of extensive computations (see the tables in section 4) we make the following conjecture:

Conjecture 1.8. The parenthesized expression in Lemma 1.7 is always an integer.

Remark 1.9. While we have stated Theorems 1.4 and 1.5 in general, in light of Theorem 1.3(c) we will henceforth (almost always) restrict our attention to $\overleftarrow{P}^n_k(x)$ for $n-k$ even.

2. Orthogonality

The key to proving orthogonality is to establish the following recursion.

Lemma 2.1. *Let $\{\overleftarrow{P}^n_k(x)\}$ be as given in Theorem 1.5.*
(a) For any $m \geq 1$, and $0 \leq j \leq m$

$$\langle x^{j+2}, \overleftarrow{P}^{j+2m}_j(x)\rangle = \langle x^{j+2m+1}, \overleftarrow{P}^{j+2m+1}_{j+1}(x)\rangle.$$

(b) For any $m \geq 1$, $0 \leq j \leq m$, and $1 \leq t \leq m-1$

$$(2m-2t)\left(\langle x^{j+2}, \overleftarrow{P}^{j+2m}_j(x)\rangle - \langle x^{j+2t+2}, \overleftarrow{P}^{j+2m}_j(x)\rangle\right) = (2t)\left(\langle x^{j+2t+1}, \overleftarrow{P}^{j+2m+1}_{j+1}(x)\rangle - \langle x^{j+2m+1}, \overleftarrow{P}^{j+2m+1}_{j+1}(x)\rangle\right).$$

Proof. For simplicity let us write

$$\overleftarrow{P}^{j+2m}_j(x) = \sum_{i=1}^{2m+1}{}' \, b_i x^{(i-1)+j},$$
$$\overleftarrow{P}^{j+2m+1}_{j+1}(x) = \sum_{i=1}^{2m+1}{}' \, c_i x^{i+j}.$$

Then, from the expressions for $w_{m,j,i}$ and $w_{m,j+1,i}$ in Theorem 1.5, we see that

$$c_i = \frac{(i+2)+2j+2m}{(i+2)+2j} b_i.$$

(a) We directly compute

$$\langle x^{j+2}, \overleftarrow{P}^{j+2m}_j(x)\rangle = 2\sum_{i=1}^{2m+1}{}' \frac{b_i}{(i+2)+2j},$$
$$\langle x^{j+2m+1}, \overleftarrow{P}^{j+2m+1}_{j+1}(x)\rangle = 2\sum_{i=1}^{2m+1}{}' \frac{c_i}{(i+2)+2j+2m},$$

and these are equal.

(b) Again we directly compute

$$(2m-2t)\left(\langle x^{j+2}, \overleftarrow{P}_j^{2m+j}(x)\rangle - \langle x^{j+2t+2}, \overleftarrow{P}_j^{2m+j}(x)\rangle\right) = 2\sum_{i=1}^{2m+1}{}' \frac{(2m-2t)(2t)}{(i+2+2j)(i+2+2j+2t)} b_i,$$

$$(2t)\left(\langle x^{j+2t+1}, \overleftarrow{P}_{j+1}^{2m+j+1}(x)\rangle - \langle x^{j+2m+1}, \overleftarrow{P}_{j+1}^{2m+j+1}(x)\rangle\right) = 2\sum_{i=1}^{2m+1}{}' \frac{(2t)(2m-2t)}{(i+2+2j+2t)(i+2+2j+2m)} c_i,$$

and these are equal. □

Theorem 2.2. *Let $\{\overleftarrow{P}_k^n(x)\}$ be as given in Theorem 1.5. Then $\{\overleftarrow{P}_k^n(x)\}_{k=0,\ldots,n}$ are pairwise mutually orthogonal.*

Proof. We proceed by induction on n. The theorem is trivially true if $n=0$.

Fix n. Then we must show that, for every k with $0 \le n-1$, $\overleftarrow{P}_k^n(x)$ is orthogonal to $\overleftarrow{P}_j^n(x)$ for every j with $k+1 \le j \le n$. We have shown this for $k=0$ in Theorem 1.4 (noting that for $k=0$ the expression in Theorem 1.5 agrees with that in Theorem 1.4). Thus we may assume $k \ge 1$.

We recall that $\overleftarrow{P}_k^n(x)$ is even if k is even and odd if k is odd, and that every power of x appearing in $\overleftarrow{P}_k^n(x)$ is between k and n. Thus it suffices to show that $\overleftarrow{P}_k^n(x)$ is orthogonal to x^{k+2t} for any t with $1 \le t \le u$ where $u=(n-k)/2$ for $n-k$ even, and $u=(n-k-1)/2$ for $n-k$ odd.

Setting $j=k-1$ and $n=j+2m+1$ in Lemma 2.1(a), we obtain

$$\langle x^{k+1}, \overleftarrow{P}_{k-1}^{n-1}(x)\rangle = \langle x^n, \overleftarrow{P}_k^n(x)\rangle \text{ for any } 1 \le k \le n-1.$$

By the inductive hypothesis, the left hand side is 0, so $\langle x^n, \overleftarrow{P}_k^n(x)\rangle = 0$.

Then, with the same substitutions, from Lemma 2.1(b), the inductive hypothesis, and our conclusion that $\langle x^n, \overleftarrow{P}_k^n(x)\rangle = 0$, we obtain

$$0 = (2t)\langle x^{k+2t}, \overleftarrow{P}_k^n(x)\rangle \text{ for any } 1 \le k \le n-1,\ 1 \le t \le u,$$

so $\langle x^{k+2t}, \overleftarrow{P}_k^n(x)\rangle = 0$, and by induction we are done. □

3. NORMALIZATION

The Legendre polynomials satisfy a three-term recurrence relation ([1, 22.1.4, 22.1.5, and 22.3.8]). We begin by deriving a three-term recurrence relation for the reverse Legendre polynomials, a result interesting in its own right.

Theorem 3.1. *Let $\{\overleftarrow{P}_k^n(x)\}$ be as given in Theorem 1.5. For any $n \ge 2$ and for any integer k with $0 \le k \le n-2$ and $n-k$ even,*

$$\overleftarrow{P}_k^n(x) = (-1)\frac{n+k+1}{n-k}\overleftarrow{P}_k^{n-2}(x) + \frac{2n+1}{n-k}x\overleftarrow{P}_{k+1}^{n-1}(x).$$

Proof. We look for a recursion of the form

$$\overleftarrow{P}^n_k(x) = \alpha \overleftarrow{P}^{n-2}_k(x) + \beta x \overleftarrow{P}^{n-1}_{k+1}(x).$$

We note that the low-order terms of $\overleftarrow{P}^n_k(x)$ and $\overleftarrow{P}^{n-2}_k(x)$ are of degree k while the low-order term of $x\overleftarrow{P}^{n-1}_{k+1}(x)$ is of degree $k+2$, and the high-order terms of $\overleftarrow{P}^n_k(x)$ and $x\overleftarrow{P}^{n-1}_{k+1}(x)$ are of degree n while the high-order term of $\overleftarrow{P}^{n-2}_k(x)$ is of degree $n-2$. Hence if there is such a recursion we must have

$$\alpha = \frac{\text{trailing coefficient of } \overleftarrow{P}^n_k(x)}{\text{trailing coefficient of } \overleftarrow{P}^{n-2}_k(x)},$$
$$\beta = \frac{\text{leading coefficient of } \overleftarrow{P}^n_k(x)}{\text{leading coefficient of } \overleftarrow{P}^{n-1}_{k+1}(x)}.$$

Referring to Theorem 1.5, we see that these ratios are

$$\alpha = \frac{w_{(n-k)/2,k,1}}{w_{(n-k-2)/2,k,1}} \cdot \frac{a_{n-k+1,1}}{a_{n-k-1,1}},$$
$$\beta = \frac{w_{(n-k)/2,k,n-k+1}}{w_{(n-k-2)/2,k+1,n-k-1}} \cdot \frac{a_{n-k+1,n-k+1}}{a_{n-k-1,n-k-1}}.$$

Substituting the expression in Lemma 1.2 and the first expression in Lemma 1.6, and doing some elementary algebra, we find that

$$\alpha = \frac{n+k+1}{n-k+1} \cdot (-1)\frac{n-k+1}{n-k} = (-1)\frac{n+k+1}{n-k},$$
$$\beta = \frac{(n-k+1)(2n+1)}{(2n-2k-1)(2n-2k+1)} \cdot \frac{(2n-2k-1)(2n-2k+1)}{(n-k)(n-k+1)} = \frac{2n+1}{n-k}.$$

Now we must establish that this relation holds for the intermediate terms, i.e, that for every $i = 1, \dots, (n-k-2/2)$ we have that

$$\text{coefficient of } x^{2i+k} \text{ in } \overleftarrow{P}^n_k(x) = \alpha \Big(\text{coefficient of } x^{2i+k} \text{ in } \overleftarrow{P}^{n-2}_k(x)\Big) + \beta \Big(\text{coefficient of } x^{2i+k-1} \text{ in } \overleftarrow{P}^{n-1}_{k+1}(x)\Big)$$

for these values of α and β.

Substituting the first expression in Lemma 1.6 and doing some elementary algebra, we find that this relation is equivalent to the following relation among coeficients of Legendre polynomials:

$$(2i+1)(2i+n+k+1)a_{n-k+1,2i+1} = \alpha(2i+1)(2i+n-k+1)a_{n-k-1,2i+1} + \beta(2i+n-k-1)(2i+n-k+1)a_{n-k-1,2i-1}$$

for $i = 1, \dots, (n-k-2)/2$. (If we adopt the convention that $a_{2m+1,j} = 0$ if $j > 2m+1$ or $j < 0$, then that will hold for $i = 0, \dots, (n-k)/2$, by our determination of α and β.)

Substituting from Lemma 1.2 and doing some more elementary algebra, we find that this relation is equivalent to

$$(n-k)(2i+n+k+1) = (n-k-2i)(n+k+1) + 2i(2n+1),$$

which is trivial to verify. □

With this recursion in hand, it is very easy to prove normalization.

Theorem 3.2. *Let $\{\overleftarrow{P}_k^n(x)\}$ be as given in Theorem 1.5. For any nonnegative integer n and any k with $0 \leq k \leq n$, $\overleftarrow{P}_k^n(1) = 1$.*

Proof. It suffices to prove this for $n-k$ even. Let $n-k = 2m$. We proceed by induction on m. By Theorem 1.4(a), $\overleftarrow{P}_n^n(1) = 1$, so the theorem is true for $m = 0$. Assume the theorem is true for $m-1$. By Theorem 3.1 and the inductive hypothesis,

$$\overleftarrow{P}_k^n(1) = (-1)\frac{n+k+1}{n-k}\overleftarrow{P}_k^{n-2}(1) + \frac{2n+1}{n-k}(1)\overleftarrow{P}_{k+1}^{n-1}(1) = (-1)\frac{n+k+1}{n-k} + \frac{2n+1}{n-k} = 1,$$

and so the theorem is true for m. Then, by induction, it is true in general. □

4. Table of values

We let $J_k^n = 2^{j(n-k)}$ where $j(0) = j(1) = 0,\ j(2) = j(3) = 1,\ j(4) = j(5) = 3,\ j(6) = j(7) = 4,\ j(8) = j(9) = 7,\ j(10) = 8.$

Table of $J_k^n \overleftarrow{P}_k^n(x)$

6							x^6
5						x^5	x^5
4					x^4	x^4	$13x^6 - 11x^4$
3				x^3	x^3	$11x^5 - 9x^3$	$11x^5 - 9x^3$
2			x^2	x^2	$9x^4 - 7x^2$	$9x^4 - 7x^2$	$143x^6 - 198x^4 + 63x^2$
1		x	x	$7x^3 - 5x$	$7x^3 - 5x$	$99x^5 - 126x^3 + 35x$	$99x^5 - 126x^3 + 35x$
0	1	1	$5x^2 - 3$	$5x^2 - 3$	$63x^4 - 70x^2 + 15$	$63x^4 - 70x^2 + 15$	$429x^6 - 693x^4 + 315x^2 - 35$
	0	1	2	3	4	5	6

8		x^8
7	x^7	x^7
6	x^6	$17x^8 - 15x^6$
5	$15x^7 - 13x^5$	$15x^7 - 13x^5$
4	$13x^6 - 11x^4$	$255x^8 - 390x^6 + 143x^4$
3	$195x^7 - 286x^5 + 99x^3$	$195x^7 - 286x^5 + 99x^3$
2	$143x^6 - 198x^4 + 63x^2$	$1105x^8 - 2145x^6 + 1287x^4 - 231x^2$
1	$715x^7 - 1287x^5 + 693x^3 - 105x$	$715x^7 - 1287x^5 + 693x^3 - 105x$
0	$429x^6 - 693x^4 + 315x^2 - 35$	$12155x^8 - 25740x^6 + 18018x^4 - 4620x^2 + 315$
	7	8

10		x^{10}
9	x^9	x^9
8	x^8	$21x^{10} - 19x^8$
7	$19x^9 - 17x^7$	$19x^9 - 17x^7$
6	$17x^8 - 15x^6$	$399x^{10} - 646x^8 + 255x^6$
5	$323x^9 - 510x^7 + 195x^5$	$323x^9 - 510x^7 + 195x^5$
4	$255x^8 - 390x^6 + 143x^4$	$2261x^{10} - 4845x^8 + 3315x^6 - 715x^4$
3	$1615x^9 - 3315x^7 + 2145x^5 - 429x^3$	$1615x^9 - 3315x^7 + 2145x^5 - 429x^3$
2	$1105x^8 - 2145x^6 + 1287x^4 - 231x^2$	$33915x^{10} - 83980x^8 + 72930x^6 - 25740x^4 + 3003x^2$
1	$20995x^9 - 48620x^7 + 38610x^5 - 12012x^3 + 1155x$	$20995x^9 - 48620x^7 + 38610x^5 - 12012x^3 + 1155x$
0	$12155x^8 - 25740x^6 + 18018x^4 - 4620x^2 + 315$	$88179x^{10} - 230945x^8 + 218790x^6 - 90090x^4 + 15015x^2 - 693$
	9	10

5. Application to quadrature

As is well known, there is an application of Legendre polynomials to quadrature. Let P_{2k-1} be the vector space of polynomials of degree at most $2k-1$, so that P_{2k-1} has dimension $2k$. Then for any polynomial $f(x)$ in P_{2k-1}, there is a formula for $\int_{-1}^{1} f(x)dx$ in terms of the values of $f(x)$ at the k zeroes of the Legendre polynomial $P_k(x)$. We derive a similar application for the reverse Legendre polynomials.

Let k and m be fixed but arbitrary. Let V be the vector space of all polynomials of degree at most $4m+2k$ that are divisible by x^{2k+1}, so that V has dimension $4m$. We derive a formula, valid for any polynomial $f(x)$ in V, for $\int_{-1}^{1} f(x)dx$ in terms of the values of $f(x)$ at the $2m$ zeroes of $\overleftarrow{P}_k^{2m+k}(x)$ other than $x=0$.

We denote the zeroes of $\overleftarrow{P}_k^{2m+k}(x)$ other than $x=0$ by $r_{-m} < \ldots < r_{-1} < r_1 < \ldots < r_m$, where $r_{-i} = -r_i$ for each i. (Recall Theorem 1.3(d).) We define functions $q_i(x)$ by

$$q_i(x) = x^{2k+2} \prod_{j \neq i} (x - r_j)^2$$

and observe that $q_i(x)$ is a polynomial of degree $4m+2k$ in V. We define constants c_i by

$$c_i = (1/q_i(r_i)) \int_{-1}^{1} q_i(x)dx$$

and observe that $c_i > 0$ and that $c_{-i} = c_i$, for each i.

Theorem 5.1. *For any polynomial $f(x)$ in V,*

$$\int_{-1}^{1} f(x)dx = \sum_i c_i f(r_i).$$

Proof. Let W be the subspace of V consisting of all polynomials of degree at most $4m+2k$ that are divisible by $x^{2k+2m+1}$. Then W is a vector space of dimension $2m$. V has basis

$$\mathscr{B} = \{x^{k+1}\overleftarrow{P}_k^{2m+k}(x), \ldots, x^{k+2m}\overleftarrow{P}_k^{2m+k}(x), x^{2k+2m+1}, \ldots, x^{2k+4m}\}$$

(as $\overleftarrow{P}_k^{2m+k}(x)$ is divisible by x^k but not x^{k+1}) and the last $2m$ vectors in this basis form a basis $\mathscr{C}$ of W.

Let V^* and W^* be the duals of V and W. For a polynomial $f(x)$ in V (resp. W), let $I(f) = \int_{-1}^{1} f(x)dx$, so that $I \in V^*$ (resp. W^*). Also, for a point r in [-1,1], let $e_r(f(x)) = f(r)$, so that $e_r \in V^*$ (resp. W^*). Then, as the points $\{r_{-m}, \ldots, r_m\}$ are nonzero and distinct, $\{e_{r_{-m}}, \ldots, e_{r_m}\}$ is linearly independent and hence a basis of W^*, so we have that

$$I = \sum_i \gamma_i e_{r_i},$$

i.e.,

$$\int_{-1}^{1} f(x)dx = \sum_i \gamma_i f(r_i)$$

for every polynomial $f(x)$ in W, for unique constants $\{\gamma_{-m}, \ldots, \gamma_m\}$. We claim that this equation holds for every polynomial $f(x)$ in V. To verify that, it suffices to show that it holds on each of the basis elements $x^t \overleftarrow{P}_k^{2m+k}(x)$, $t = k+1, \ldots, k+2m$. But the left hand side of this equation is

$$\int_{-1}^{1} x^t \overleftarrow{P}_k^{2m+k}(x) = \langle x^t, \overleftarrow{P}_k^{2m+k}(x) \rangle = 0$$

as $\overleftarrow{P}_k^{2m+k}(x)$ is orthogonal to any such polynomial x^t, while the right hand side of this equation is

$$\sum_i \gamma_i (r_i)^t \overleftarrow{P}_k^{2m+k}(r_i) = 0$$

precisely because the r_i are the roots of $\overleftarrow{P}_k^{2m+k}(x)$.

It remains to identify the constants γ_i. To this end, consider the set of functions

$$\mathscr{D} = \{q_{-m}(x), \ldots, q_m(x)\}.$$

For each function in this set we have

$$\int_{-1}^{1} q_i(x)dx = \sum_j \gamma_j q_i(r_j) = \gamma_i q_i(r_i)$$

as $q_i(r_j) = 0$ for $j \neq i$, yielding $\gamma_i = c_i$ as above. □

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[1]

Right Noetherian ring in which every ideal is prime.

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Abstract

A ring in which every ideal is prime (*R* is a prime ring and every proper ideal is a prime ideal) is called a fully prime ring. A ring *R* is fully prime if and only if every ideal is idempotent and the set of ideals is linearly ordered. The structure of such rings was studied and several examples are given in Blair-Tsutsui [1]. The structures of rings which are related to fully prime rings were also studied, for example, in Tsutsui [8] and Hirano [4]. It had been an open problem since 1993, noted on Blair-Tsutsui [1], to determine if a right Noetherian fully prime ring is primitive. In this paper, we solve this problem affirmatively.

Throughout this paper a ring *R* is with identity. For a non-empty collection $\underline{A}$ of right ideals of *R*, a right *R*-module *X* is called *A*-injective if every *R*-map $\phi : A \to X$ with $A \in \underline{A}$ can be lifted to $\overline{\phi} : R \to X$. For a right ideal *I*, we denote $I_2 = \{r \in R \mid Rr \subseteq I\}$, the unique largest (two-sided) ideal contained in *I*. For any ring *T*, we let *S*(*T*) denote the correction of (prime) ideals of *T*.

Lemma (Lemma 1 of P.F. Smith [7]): Let $\underline{A}$ be a nonempty collection of right ideals of *R* with $R \in \underline{A}$, and *X* an $\underline{A}$-injective right *R*-module. Let *I* be an ideal of *R* and $\tilde{\underline{A}}$ the collection of right ideals *A*/*I* of *R*/*I* with $A \in \underline{A}$, and $A \supseteq I$. Let $Y = \{x \in X \mid xI = 0\}$. Then *Y* is an $\tilde{\underline{A}}$-injective right *R*/*I*-module.

Theorem: Every factor ring of a right Noetherian fully prime ring *R* is primitive.

Sketch of Proof. It is enough to show that 0 is a primitive ideal for every factor ring of a right Noetherian fully prime ring is right Noetherian and fully prime.

Suppose that *R* is not primitive and every *S*(*R*)-injective right *R*-module is not injective.

[1] A part of the contents with further details may be submitted to elsewhere as a part of our on-going work.

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Let G be the set of ideals P of R such that not every $S(R/P)$-injective right R/P-module is injective. Then G is non-empty since $0 \in G$. Because R is right Noetherian, we may choose maximal element P in G. Further, without loss of generality, we may assume that $P = 0$.

Let X be a right R-module which is $S(R)$ injective but not injective. Then there exists (by Baer criterion) a right ideal E of R, and an R-map $\phi : E \to X$ such that ϕ cannot be lifted to R.

Because R is right Noetherian, we may choose E to be maximal with respect to the property contained in RE.

If $E_2 \neq 0$, we will reach a contradiction.

If $E_2 = 0$, then we let $\overline{E}$ be the right ideal containing E and maximal with respect to properly contained in RE. Suppose that $\overline{E}_2 = 0$. Then $RE / \overline{E}$ is a simple R-module. If for any $r \in R$, $REr \in \overline{E}$, then $(RE)(RrR) \subseteq \overline{E}$, and since the set of ideals of a fully prime ring is linearly ordered, and every ideal is idempotent, either $(RE)(RrR) = (RE)$, or $(RE)(RrR) = RrR$. As $\overline{E} \subset RE$, we have $RrR \subset E$, and as $RrR \subseteq \overline{E} = 0$, $RE / \overline{E}$ is faithful and this implies R is primitive. Thus, we now assume $\overline{E}_2 \neq 0$, and this will get a conclusion that R is either primitive or every $S(R)$-injective right R-module is injective.

Suppose that $S(R)$-injective right R-module is injective. In this case we see that R is a simple ring and hence primitive: Suppose that R is not simple. Consider R/M where M is the nonzero maximal ideal of R. Since every ideal of R is idempotent and the set of ideals is linearly ordered, for any ideal P, if $\phi : P \to R / M$ is a R-map, $\phi(P) = \phi(P)P = 0$, and hence R/M is $S(R)$-injective and therefore it is injective. On the other hand, since R is prime Noetherian, M contains a regular element a. Since R/M is injective, the lifting property of the map $\phi : aR \to R/M$ by $\phi(ar) = r + M$ implies $\phi(a) = 1 + M = \overline{\phi}(1)a = M$, a contradiction.

By Theorem 3.4 of Blair-Tsutsui [1], a right Noetherian fully right bounded ring is simple Artinian. We now have the following:

Corollary 1: Let R be a right Noetherian fully prime ring. Then the following is equivalent:

(a) R is right bounded.
(b) R is fully right bounded.
(c) R is simple Artinian.

Corollary 2: Let R be a right Noetherian fully prime ring with an ideal P. Then R/P is right bounded if and only if R/P is simple Artinian.

Example 1: Theorem 2 of Hirano [4] gives a fully (completely) prime ring (domain) with n ideals. For a field of k of characteristic 0, if we let $D = A_1(k)$, the Weyl algebra with char, then R_1 is a fully prime Noetherian domain with exactly one non-zero proper ideal $xA_1(k)$ as shown in Theorem 4.6 of Blair-Tsutsui [1]. Since R_n is the idealizer of $T_k^n(D) \cong A_n(k)$, and $A_n(k)$ is a simple right Noetherian domain, R_n is a fully prime right Noetherian ring with exactly n nonzero ideals.

We remark that while the ring R_n in Example 1 is a domain, a fully prime right Noetherian ring is not necessarily a domain as a simple Noetherian ring that is not a domain exists (See Example 14.17 of Chatters-Hajarnavis [2]). Note also that a prime right Noetherian ring is simple Artinian if Soc(R) $\neq 0$ (See Theorem 1.24 of Chatters-Hajarnavis [2]). Thus Soc(R) of the ring R_n in Example 1 is zero, or more generally, a non-simple fully prime right Noetherian ring R is a nonsingular primitive ring with $\mathrm{Soc}(R) = 0$.

Proposition. A fully prime ring R is either a nonsingular primitive ring with Soc(R) being the minimum nonzero (two sided) ideal, or a ring with $\mathrm{Soc}(R) = 0$.

Proof: If $\mathrm{Soc}(R) \neq 0$, then, since R is prime, R is primitive. Further, since Soc(R) is the intersection of all essential right ideals; all ideals in a prime ring is essential; and every ideal of a fully prime ring is linearly ordered; we have $0 \neq \mathrm{Soc}(R) \subseteq \bigcap_{0 \neq P_i \triangleleft R} P_i$. Thus, Soc(R) is the minimum nonzero ideal of R. As $\mathrm{Sing}(R) \cdot \mathrm{Soc}(R) = 0$, $\mathrm{Sing}(R) = 0$.

Note that since a fully prime right Noetherian ring is in particular a strongly prime ring, it is nonsingular by Proposition II.1 of Handelman-Lawrence [3].

Example 2. A ring R that is not nonsingular is given in (11.21) Example of Lam [6]. R has exactly one nonzero proper ideal and the ideal is idempotent. Hence R is a fully prime ring that is not nonsingular.

The next example shows that exists a non-primitive fully prime ring (with identity) that has infinitely many ideals and the intersection of nonzero ideals is nonzero.

Example 3.
Mazurek-Roszkowska [5] constructed an example of a chain ring S without identity: S is a domain whose lattice of left ideals as well as the lattice of right ideals are linearly ordered such that
(1) S is a F algebra and every ideal of S is a F ideal,
(2) $J(S) = S$,
(3) $S/I \approx S$ for each ideal I of S (hence I is a completely prime ideal), and
(4) S has countably many ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_n \cdots \subset I_\omega = S$.
We merely embed S into a ring R with identity as the standard way:
Let $R = S \oplus F$ where addition is defined component wise and multiplication is given by $(s_1, k_1)(s_2, k_2) = (s_1 s_2 + k_1 s_2 + k_2 s_1, s_1 s_2)$ where $s_1, s_2 \in S,\ k_1, k_2 \in F$.
Let $M = S \oplus 0$. Then as $R/M \approx F$, M is a maximal right ideal. Hence $J(R) = M \cap J(R)$. But then, since $J(S) = S$, $J(M) = M = M \cap J(R) = J(R)$. As $M \neq 0$, R is not semiprimitive. As $M = J(R)$, M is a unique maximal right ideal of R, and hence every right ideal of R is contained in M. If $T \oplus 0 \subseteq M$ is an ideal of R, then, T is an ideal of S. On the other hand, for any for any ideal I of S, let $(i, 0) \in I \oplus 0$. then $(i,0)(s,k) = (is + ki, 0)$, $(s,k)(i,0) = (si + ki, 0)$ for any $(s,k) \in R$. As I is an F algebra, this shows that only (two sided) ideals of R are of the form $I \oplus 0$ for ideal I of S.

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