Proceedings of the Exchange of Mathematical Ideas

2018

Exchange of Mathematical Ideas - EMI 2018
Proceedings of the Sixth Annual Conference
of the Exchange of Mathematical Ideas
Prescott, Arizona
July 20–21, 2018

Organizers

Douglas Mupasiri, University of Northern Iowa
Hisa Tsutsui, Embry-Riddle Aeronautical University

Conference Executive Committee

Ed Poon, Embry-Riddle Aeronautical University
Brent Solie, Embry-Riddle Aeronautical University
Chris Briggs, Embry-Riddle Aeronautical University

Editor

Chris Briggs, Embry-Riddle Aeronautical University

ACKNOWLEDGMENT OF SUPPORT: The organizers gratefully acknowledge the support of their respective institutions for this conference. We also gratefully acknowledge the generous support of Tsunekazu Nishinaka (University of Hyogo) under Grant-in Aid for Scientific Research (C) 17K05207, JSPS.

ISBN: 978-1-09-522461-8 DOI of this front matter: 10.5281/zenodo.2593813

FOREWORD

The International Conference for the Exchange of Mathematical Ideas is the brainchild of three founding organizers: Douglas Mupasiri of University of Northern Iowa, Keith Mellinger of University of Mary Washington, and Hisa Tsutsui of Embry-Riddle Aeronautical University. The first conference took place at Embry-Riddle Aeronautical University's Prescott campus on May 26, 2012. It had an international audience of 21 participants representing diverse mathematical specialties ranging from noncommutative ring theory to computability theory, cryptography to topology, algebraic number theory to operator theory.

The ethos of the conference is grounded on recognition of the surprising connections that arise between distant fields. That by getting together to describe their research to an audience of nonspecialists, researchers often gain new perspective on their own work, and find inspiration in the work of others. The EMI is intended to provide a venue for mathematicians to interact in this way. Indeed, collaborations across disciplines sparked at the Exchange have resulted in research productivity, including peer-review journal publications.

Most of all, even though mathematics can be done alone, and often is the product of individual effort, it gains meaning only when shared. We gather to pay homage to this communal aspect of mathematics. We dedicate these proceedings to those who have been with us in the past, and those who will join us in the future.

Contents

Infinite groups and primitivity of their group rings
Tsunekazu Nishinaka 5

Semigroup rings over semiprime ring semigroups
Hisaya Tsutsui 20

Separable polynomials in skew polynomials rings
Satoshi Yamanaka 25

Internal Bousfield Localizations
Renaud Gauthier 36

Models of Discrete Conformal Geometry
William E. Wood 73

Positive solutions to some systems of nonlinear Diophantine equations
Christopher Briggs 80

Multiplicative sets of idempotents in a semilocal ring
Yasuyuki Hirano 86

A Toronto Space
Philip Stephen Doi 90

Mathematical Aspects of Kolam, a Dravidian Art Form
R. Kit Kittappa 95

Introduction to Graph Pebbling and Rubbling
Nándor Sieben 111

A useful homomorphic image of a free abelian by infinite cyclic group
Christopher Briggs 122

Infinite groups and primitivity of their group rings

Tsunekazu Nishinaka *
Department of Business Administration
Okayama Shoka University

A ring R is (right) primitive provided it has a faithful irreducible (right) R-module. If non-trivial group G is finite or abelian, then the group algebra KG over a field K cannot be primitive. If G has non-abelian free subgroups, then KG is often primitive. In the present note, we focus on a local property which is often satisfied by groups with non-abelian free subgroups:

(∗) For each subset M of G consisting of finite number of elements not equal to 1, there exist three distinct elements a, b, c in G such that whenever $x_i \in \{a, b, c\}$ and $(x_1^{-1} g_1 x_1) \cdots (x_m^{-1} g_m x_m) = 1$ for some $g_i \in M$, $x_i = x_{i+1}$ for some i.

We can see that the group algebra KG of a group G over a field K is primitive provided G has a free subgroup with the same cardinality as G and satisfies (∗). In particular, for every countably infinite group G satisfying (∗), KG is primitive for any field K. As an application of this theorem, we can see primitivity of group algebras of many kinds of groups with non-abelian free subgroups which includes a recent result; the primitivity of group algebras of one relator groups with torsion.

1 A brief history of the research

Let R be a ring with the identity element. It need not to be commutative. A ring R is right primitive if and only if there exists a faithful irreducible right R-module M_R, where M_R is irreducible provided it has no non-trivial submodules, and M_R is faithful provided the annihilator of M is zero: $ann(M_R) = \{r \in R \mid Mr = 0\} = 0$. The above definition of primitivity is equivalent to the following: A ring R is right primitive if and only if there exists a maximal right ideal ρ which contains no non-trivial ideals of R. A left primitive ring is similarly defined. In what follows, for right primitive, we simply call it primitive. Speaking of a group ring, a right primitive group ring is always left primitive. In this section, we introduce briefly a history of the research to primitivity of group rings.

Since the group ring KG of a non-trivial group G over a field K has always the augmentation ideal which is non-trivial, it cannot be a simple ring. If G is a finite group, then KG is a finite dimensional algebra and so it is never primitive because a finite dimensional algebra is simple provided it is primitive. Moreover,

*Supported by Grants-in-Aid for Scientific Research under grant no. 23540063 and 17K05207. This work was also supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

 10.5281/zenodo.2579471

if a commutative ring is primitive, then it is a field, and therefore if $G \neq 1$ is abelian, then KG is never primitive. Hence primitivity of KG is appeared only in the case that G is non-abelian and non-finite. For the longest time no examples of primitive group rings were known, and it was thought that KG could not be primitive provided $G \neq 1$.

The first example of primitive group rings was offered by Formanek and Snider [7] in 1972, and in 1973 Formanek [6] gave the primitivity of group rings of well-known groups; namely the primitivity of group rings of free products.

Theorem 1.1. (Formanek[6]) *Let G be a free product of non-trivial groups (except $G = \mathbb{Z}_2 * \mathbb{Z}_2$); Then KG is primitive for any field K.*

In particular, if G is a free group then KG is primitive for any field K. After that, many examples of primitive group rings were constructed. In 1978, Domanov [4], Farkas-Passman [5] and Roseblade [17] gave the complete solution for primitivity of group rings of polycyclic-by-finite groups.

Theorem 1.2. (Domanov[4], Farkas-Passman[5],Roseblade[17]) *Let G be a non-trivial polycyclic-by-finite group. Then KG is primitive if and only if $\Delta(G) = 1$ and K is non-absolute, where $\Delta(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$ and K is absolute if it is algebraic over a finite field.*

Polycyclic-by-finite groups are belong to the class of noetherian groups, and it is not easy to find a noetherian group which is not polycyclic-by-finite [15]. Therefore almost all other known infinite groups belong to the class of non-noetherian groups. As is well known, if KG is noetherian then G is also noetherian, but the converse is not true generally. A group of the class of finitely generated non-noetherian groups has often non-abelian free subgroups; for instance, a free group, a locally free group, a free product, an amalgamated free product, an HNN-extension, a Fuchsian group, a one relator group, etc. It is known that a free Burnside group is not the case, though. After the result Theorem 1.1 above, primitivity of group rings of known groups which are non-noetherian has been obtained gradually. Theorem 1.1 was generalized to one for amalgamated free products by Balogun in 1989:

Theorem 1.3. ([1, Balogun, '89]) *Let $G = A *_H B$ be the free product of A and B with H amalgamated. If there exist elements $a \in A \setminus H$ and $b \in B \setminus H$ such that $a^2, b^2 \notin H$, $a^{-1}Ha \cap H = 1$ and $b^{-1}Hb \cap H = 1$, then KG is primitive for any field K.*

In 1997, the primitivity of semigroup algebras of free products was given by Chaudhry, Crabb and McGregor [2].

The primitivity of a group ring of a free group F extended to one for the ascending HNN extension $G = F_\varphi$ of a free group F; for the case of $|F| = \aleph_0$ in 2007 and for the case of arbitrary cardinality of F in 2011:

Theorem 1.4. ([13, Nishinaka, '07], [14, Nishinaka, '11]) *Let F be a non-abelian free group, and $G = F_\varphi$ the ascending HNN extension of F determined by φ. Then the following are equivalent:*

(1) *KG is primitive for a field K.*

(2) *$|K| \leq |F|$ or G is not virtually the direct product $F \times \mathbb{Z}$.*

(3) *$|K| \leq |F|$ or $\triangle(G) = 1$.*

In particular, if G is a strictly ascending HNN extension, that is, $\varphi(F) \neq F$, then KG is primitive for any field K.

Moreover, the primitivity of group rings of free groups extended to one for locally free groups:

Theorem 1.5. ([14, Nishinaka, '11]) *Let G be a non-abelian locally free group which has a free subgroup whose cardinality is the same as that of G itself. If K is a field then KG is primitive.*

In particular, every group ring of the union of an ascending sequence of non-abelian free groups over a field is primitive, and so every group ring of a countable non-abelian locally free group over a field is primitive.

Now, there is no viable conjecture as to when KG is primitive for arbitrary groups. There exists a non-primitive KG for any field K even in the case that KG is semiprimitive and $\Delta(G) = 1$ (See [3]).

2 Group algebras of groups with free subgroups

In the present note, we focus on a local property which is often satisfied by groups with non-abelian free subgroups:

($*$) For each subset M of G consisting of finite number of elements not equal to 1, there exist three distinct elements a, b, c in G such that whenever $x_i \in \{a, b, c\}$ and $(x_1^{-1}g_1x_1)\cdots(x_m^{-1}g_mx_m) = 1$ for some $g_i \in M$, $x_i = x_{i+1}$ for some i.

We can see that if G is countably infinite group and satisfies ($*$), then KG is primitive for any field K. More generally, we can get the following theorem:

Theorem 2.1. *Let G be a non-trivial group which has a free subgroup whose cardinality is the same as that of G. Suppose that G satisfies the condition ($*$). If R is a domain with $|R| \leq |G|$, then the group ring RG of G over R is primitive.*

In particular, the group algebra KG is primitive for any field K.

As an application of the theorem, we give the primitivity of group algebras of one relator groups with torsion:

Theorem 2.2. *If G is a non-cyclic one relator group with torsion, then KG is primitive for any field K.*

One of the main method to prove Theorem 2.1 is a graph theoretic method which is called SR-graph theory.

3 SR-graph theory

Let $\mathcal{G} = (V, E)$ denote a simple graph; a finite undirected graph which has no multiple edges or loops, where V is the set of vertices and E is the set of edges. A finite sequence $v_0e_1v_1 \cdots e_pv_p$ whose terms are alternately elements e_q's in E and v_q's in V is called a path of length p in $\mathcal{G}$ if $v_q \neq v_{q'}$ for any $q, q' \in \{0, 1, \cdots, p\}$ with $q \neq q'$; it is often simply denoted by $v_0v_1 \cdots v_p$. Two vertices v and w of $\mathcal{G}$ are said to be connected if there exists a path from v to w in $\mathcal{G}$. Connection is an equivalence relation on V, and so there exists a decomposition of V into subsets C_i's $(1 \leq i \leq m)$ for some $m > 0$ such that $v, w \in V$ are connected if and only if both v and w belong to the same set C_i. The subgraph (C_i, E_i) of $\mathcal{G}$ generated by C_i is called a (connected) component of $\mathcal{G}$. Any graph is a disjoint union of components. For $v \in V$, we denote by $C(v)$ the component of $\mathcal{G}$ which contains the vertex v.

We define a graph which has two distinct edge sets E and F on the same vertex set V. We call such a triple (V, E, F) an SR-graph provided that $(V, E \cup F)$ is a simple graph (i.e. a finite undirected graph which has no multiple edges or loops) and every component of the graph (V, E) is a complete graph (see Fig 1 and Fig 2). That is, we define an SR-graph as follows:

Definition 3.1. *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$ be simple graphs with the same vertex set V. For $v \in V$, let $U(v)$ be the set consisting of all neighbours of v in $\mathcal{H}$ and v itself: $U(v) = \{w \in V \mid vw \in F\} \cup \{v\}$. A triple (V, E, F) is an SR-graph (for a sprint relay like graph) if it satisfies the following conditions*:

(SR1) *For any $v \in V$, $C(v) \cap U(v) = \{v\}$.*
(SR2) *Every component of $\mathcal{G}$ is a complete graph.*

If $\mathcal{G}$ has no isolated vertices, that is, if $v \in V$ then $vw \in E$ for some $w \in V$, then SR-graph (V, E, F) is called a proper SR-graph.

We call $U(v)$ the SR-neighbour set of $v \in V$, and set $\mathfrak{U}(V) = \{U(v) \mid v \in V\}$. For $v, w \in V$ with $v \neq w$, it may happen that $U(v) = U(w)$, and so $|\mathfrak{U}(V)| \leq |V|$ generally. Let $\mathcal{S} = (V, E, F)$ be an SR-graph. We say $\mathcal{S}$ is connected if the graph $(V, E \cup F)$ is connected.

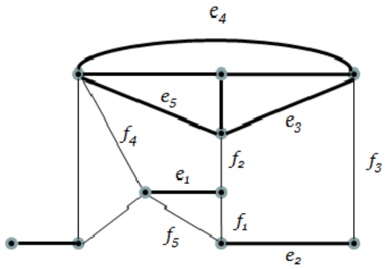


Fig 1. An example of an SR-graph: bold solid lines are edges in E and normal solid lines are edges in F. Sequences $(e_1, f_1, e_2, f_3, e_4, f_4)$, $(e_1, f_2, e_3, f_3, e_2, f_5)$ and (e_1, f_2, e_5, f_4) are SR-cycles.

Fig 2. Prohibits : It is not allowed to exist the above subgraph in an SR-graph.

Definition 3.2. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $p > 1$. Then a path $v_1w_1v_2w_2, \cdots, v_pw_pv_{p+1}$ in the graph $(V, E \cup F)$ is called a SR-path of length p in $\mathcal{S}$ if either $e_q = v_qw_q \in E$ and $f_q = w_qv_{q+1} \in F$ or $f_q = v_qw_q \in F$ and $e_q = w_qv_{q+1} \in E$ for $1 \leq q \leq p$; simply denoted by $(e_1, f_1, \cdots, e_p, f_p)$ or $(f_1, e_1, \cdots, f_p, e_p)$, respectively. If, in addition, it is a cycle in $(V, E \cup F)$; namely, $v_{p+1} = v_1$, then it is an SR-cycle of length p in $\mathcal{S}$.*

To prove Theorem 2.1, we use some results for SR-graphs and apply them to the Formanek's method. We can give Formanek's method, as follows:

Proposition 3.3. (See [6]) *Let RG be the group ring of a group G over a ring R with identity. If for each non-zero $a \in RG$, there exists an element $\varepsilon(a)$ in the ideal $RGaRG$ generated by a such that the right ideal $\rho = \sum_{a \in RG \setminus \{0\}} (\varepsilon(a) + 1)RG$ is proper; namely, $\rho \neq RG$, then RG is primitive.*

The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make ρ be proper. Now, ρ is proper if and only if $r \neq 1$ for all $r \in \rho$. Since ρ is generated by the elements of form $(\varepsilon(a) + 1)$ with $a \neq 0$, r has the presentation, $r = \sum_{(a,b) \in \Pi} (\varepsilon(a) + 1)b$, where Π is a subset which consists of finite number of elements of $RG \times RG$ both of whose components are non-zero. Moreover, $\varepsilon(a)$ and b are linear combinations of elements of G, and so we have

$$r = \sum_{(a,b) \in \Pi} \sum_{g \in S_a, h \in T_b} (\alpha_g \beta_h gh + \beta_h h), \tag{1}$$

where S_a and T_b are the support of $\varepsilon(a)$ and b respectively and both α_g and β_h are elements in K. In the above presentation (1), if there exists gh such that $gh \neq 1$ and does not coincide with the other $g'h'$'s and h''s, then $r \neq 1$ holds. Strictly speaking: Let $\Omega_{ab} = S_a \times T_b$. If there exist $(a,b) \in \Pi$ and (g,h) in Ω_{ab} with $gh \neq 1$ such that $gh \neq g'h'$ and $gh \neq h'$ for any $(c,d) \in \Pi$ and for any (g',h') in Ω_{cd} with $(g',h') \neq (g,h)$, then $r \neq 1$ holds.

On the contrary, if $r = 1$, then for each gh in (1) with $gh \neq 1$, there exists another $g'h'$ or h' in (1) such that either $gh = g'h'$ or $gh = h'$ holds. Suppose here that there exist (g_{2i-1}, h_i) and (g_{2i}, h_{i+1}) $(i = 1, \cdots, m)$ in $V = \bigcup_{(a,b)\in\Pi} \Omega_{ab} \cup T_b$ such that the following equations hold:

$$\begin{array}{llll} g_1h_1 = & g_2h_2, & & \\ & g_3h_2 = & g_4h_3, & \\ & & \ddots & \\ & & g_{2m-1}h_m = g_{2m}h_{m+1} & \text{and} \quad h_{m+1} = h_1. \end{array} \tag{2}$$

Eliminating h_i's in the above, we can see that these equations imply the equation $g_1^{-1}g_2 \cdots g_{2m-1}^{-1}g_{2m} = 1$. If we can choose $\varepsilon(a)$'s so that their supports g_i's never satisfy such an equation, then we can prove that $r \neq 1$ holds by contradiction. We need therefore only to see when supports g's of $\varepsilon(a)$'s satisfy equations as described in (2).

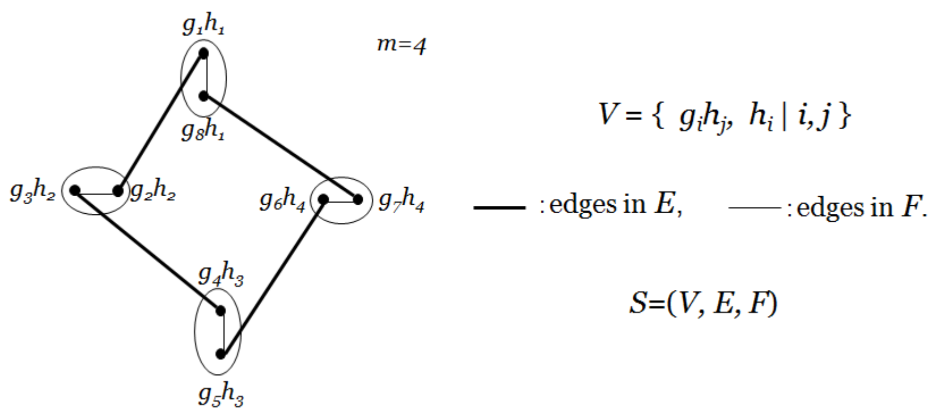


Fig 3. Equations as described in (2) for m=4.

Roughly speaking, we regard V above as the set of vertices and for $v = (g,h)$ and $w = (g',h')$ in V, we take an element vw as an edge in E provided $gh = g'h'$ in G, and take vw as an edge in F provided $g \neq g'$ and $h = h'$ in G (see Fig 3). In this situation, if there exists an SR-cycle $v_1w_1v_2w_2, \cdots, v_pw_pv_1$ in the SR-graph (V,E,F) whose adjacent terms are alternately elements v_iw_i in E and w_iv_{i+1} in F, then there exist (g_i,h_j)'s in V satisfying the desired equations as described in (2). Thus the problem can be reduced to find an SR-cycle in a given SR-graph.

By making use of graph theoretic considerations, we can prove the following theorems:

Theorem 3.4. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and let ω_E and ω_F be, respectively, the number of components of $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$. Suppose that every component of $\mathcal{H} = (V, F)$ is a complete graph and $\mathcal{S}$ is connected. Then $\mathcal{S}$ has an SR-cycle if and only if $\omega_E + \omega_F < |V| + 1$.*

In particular, if $\mathcal{S}$ is proper and $\alpha \leq \gamma$ then $\mathcal{S}$ has an SR-cycle.

Theorem 3.5. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $\mathfrak{C}(V) = \{V_1, \cdots, V_n\}$ with $n > 0$. Suppose that every component $\mathcal{H}_i = (V_i, F_i)$ of $\mathcal{H}$ is a complete k-partite graph with $k > 1$, where k is depend on $\mathcal{H}_i$. If $|V_i| > 2\mu(\mathcal{H}_i)$ for each $i \in \{1, \cdots, n\}$ and $|I_{\mathcal{G}}(V)| \leq n$ then $\mathcal{S}$ has an SR-cycle.*

4 Proof of Theorem 2.1

Let G be a group and $M_1, \cdots, M_n$ non-empty subsets of G which do not include the identity element. We say $M_1, \cdots, M_n$ are mutually reduced in G if for each finite elements $g_1, \cdots, g_m$ in the union of M_i's, $g_1 \cdots g_m = 1$ implies both g_i and g_{i+1} are in the same M_j for some i and j. If $M_1 = \{x_1^{\pm 1}\}, \cdots, M_m = \{x_m^{\pm 1}\}$ and they are mutually reduced, then we say simply $x_1, \cdots, x_m$ are mutually reduced.

In this section, we shall prove Theorem 2.1 after preparing three lemmas.

Lemma 4.1. (See [16, Theorem 2]) *Let K' be a field and G a group. If $\triangle(G)$ is trivial and $K'G$ is primitive, then for any field extension K of K', KG is primitive.*

By making use of Theorem 3.4 and Theorem 3.5, we can get the next two lemmas:

Lemma 4.2. *Let G be a non-trivial group, $m > 0$ and $n > 0$. For non-trivial distinct elements f_{ij}'s ($i = 1, 2, 3,\ j = 1, \cdots, m$) in G and for distinct elements g_i's ($i = 1, \cdots, n$) in G, we set*

$$\begin{array}{lll} S & = \bigcup_{i=1}^{3} S_i, & \text{where } S_i = \{f_{ij} \mid 1 \leq j \leq m\}, \\ T & = \{g_i \mid 1 \leq i \leq n\}, & \\ V & = S \times T, & \\ M_i & = \{f_{ij}^{\pm 1},\ f_{ij}^{-1} f_{ik} \mid j, k = 1, 2, \cdots, m,\ j \neq k\} & (i = 1, 2, 3), \\ I & = \{(f, g) \in V \mid fg \neq f'g' & \text{for any } (f', g') \in V \text{ with } (f', g') \neq (f, g)\}. \end{array}$$

Then if M_1, M_2 and M_3 are mutually reduced, then $|I| > n$.

Lemma 4.3. *Let G be a non-trivial group and $n > 0$. For each $i = 1, 2, \cdots, n$, let $f_{i1}, \cdots, f_{im_i}$ be distinct $m_i > 0$ elements of G; $f_{ip} \neq f_{iq}$ for $p \neq q$, and let x_{ij} $(1 \leq i \leq n, 1 \leq j \leq 3)$ be distinct elements in G. we set*

$$\begin{array}{ll} S & = \bigcup_{i=1}^{3} S_i, \text{ where } S_i = \{f_{ij} \mid 1 \leq j \leq m_i\}, \\ X & = \bigcup_{i=1}^{n} X_i, \text{ where } X_i = \{x_{ij} \mid 1 \leq j \leq 3\}, \\ V & = \bigcup_{i=1}^{n} V_i, \text{ where } V_i = X_i \times S_i, \\ I & = \{(x, f) \in V \mid xf \neq x'f' \text{ for any } (x', f') \in V \text{ with } (x', f') \neq (x, f)\}. \end{array}$$

If x_{ij}'s are mutually reduced elements, then $|I| > m$, where $m = m_1 + \cdots + m_n$.

Proof of Theorem 2.1. Let B be the basis of a free subgroup of G whose cardinality is the same as that of G. Then we may assume that the cardinality of B is also same as G, that is, $|B| = |G|$. In addition, since $|R| \leq |G|$, we have that $|B| = |RG|$. We can divide B into three subsets B_1, B_2 and B_3 each of whose cardinality is $|B|$. It is then obvious that the elements in B are mutually reduced. Let φ be a bijection from B to $RG \setminus \{0\}$ and σ_s a bijection from B to B_s, $s = 1, 2, 3$.

For $b \in B$, let $\varphi(b) = \sum_{f \in F_b} \alpha_f f$, where $\alpha_f \in R$ and F_b is the support of $\varphi(b)$. We set

$$M_b = \{f^{\pm 1},\ f^{-1}f' \mid f, f' \in F_b, f \neq f'\}.$$

Since G satisfies the condition $(*)$, there exist $x_{b1}, x_{b2}, x_{b3} \in G$ such that $M_b^{x_{bt}} = \{x_{bt}^{-1} f^{\pm 1} x_{bt},\ x_{bt}^{-1} f^{-1} f' x_{bt} \mid f, f' \in F_b,\ f \neq f'\}$ $(t = 1, 2, 3)$ are mutually reduced. We here define $\varepsilon(b)$ and $\varepsilon^1(b)$ by

$$\varepsilon(b) = \sum_{s=1}^{3} \sum_{t=1}^{3} \sigma_s(b) x_{bt}^{-1} \varphi(b) x_{bt} \quad \text{and} \quad \varepsilon^1(b) = \varepsilon(b) + 1. \tag{3}$$

Note that $\varepsilon(b)$ is an element in the ideal of RG generated by $\varphi(b)$. Let $\rho = \sum_{b \in B} \varepsilon^1(b) RG$ be the right ideal generated by $\varepsilon^1(b)$ for all $b \in B$. If $w \in \rho$, then we can express w by

$$w = \sum_{b \in A} \varepsilon^1(b) u_b = \sum_{b \in A} (\varepsilon(b) u_b + u_b) \tag{4}$$

for some non-empty finite subsets A of B and u_b in RG. In view of Proposition 3.3, in order to prove that RG is primitive, we need only show that ρ is proper; $\rho \neq RG$. To do this, it suffices to show that $w \neq 1$.

Let $u_b = \sum_{h \in H_b} \beta_h h$, where H_b is the support of u_b. Substituting this and $\varphi(b) = \sum_{f \in F_b} \alpha_f f$ into (3), we obtain the following expression of $\varepsilon(b) u_b$:

$$\varepsilon(b) u_b = \sum_{s=1}^{3} \sum_{t=1}^{3} \sum_{f \in F_b} \sum_{h \in H_b} \alpha_f \beta_h y_{bs} x_{bt}^{-1} f x_{bt} h, \text{ where } y_{bs} = \sigma_s(b). \tag{5}$$

In what follows, for the sake of convenience, we represent $y_{bs}x_{bt}^{-1}fx_{bt}h$ by $y_sx_t^{-1}fx_th$, and we note that y_s and x_t are depend on $b \in B$. For $s = 1, 2, 3$, we here set

$$E_{bs} = \sum_{t=1}^{3} \sum_{f \in F_b} \sum_{h \in H_b} \alpha_f \beta_h y_s \xi(x_t, f, h), \text{ where } \xi(x_t, f, h) = x_t^{-1}fx_th. \tag{6}$$

That is, $\varepsilon(b)u_b = E_{b1} + E_{b2} + E_{b3}$. We can see that there exist more than $|H_b|$ isolated elements in the expression (6) of E_{bs} for each $s = 1, 2, 3$. Strictly speaking, if we set $X_b = \{x_1, x_2, x_3\}$, $\Gamma_b = X_b \times F_b \times H_b$ and

$$\begin{aligned} I_s = \{(x_t, f, h) \quad & | \ (x_t, f, h) \in \Gamma_b,\ \xi(x_t, f, h) \neq \xi(x_p, f', h') \\ & \text{for any } (x_p, f', h') \in \Gamma_b \text{ with } (x_p, f', h') \neq (x_t, f, h)\}, \end{aligned}$$

then $|I_s| > |H_b|$. In fact, since $M_b^{x_{bt}}$ $(t = 1, 2, 3)$ are mutually reduced, it follows from lemma 4.2 that $|I_s| > |H_b|$.

Now, we shall see that $w \neq 1$ holds, where w as in (4). In (4), we set that $w_1 = \sum_{b \in A} \varepsilon(b)u_b$ and $w_2 = \sum_{b \in A} u_b$. We have then that

$$w_1 = \sum_{b \in A} \sum_{s=1}^{3} E_{bs} \quad \text{and} \quad w = w_1 + w_2.$$

Let $Supp(E_{bs})$ be the support of E_{bs} and $m_b = |Supp(E_{b1})|$. We should note that $|Supp(E_{bs})| = m_b$ for all $s = 1, 2, 3$. It is obvious that $m_b \geq |I_s|$, and so $m_b > |H_b|$ by the above. Since y_{bs} $(b \in A, 1 \leq s \leq 3)$ are mutually reduced, by virtue of Lemma 4.3, we have $|Supp(w_1)| > \sum_{b \in A} m_b$. Moreover we have that

$$\begin{aligned} |Supp(w)| \ & \geq |Supp(w_1)| - |Supp(w_2)| \\ & > \sum_{b \in A} m_b - \sum_{b \in A} |H_b| \\ & > 0, \end{aligned}$$

which implies $|Supp(w)| \geq 2$. In particular, $w \neq 1$. We have thus seen that RG is primitive.

Finally, we shall show that KG is primitive for any field K. Let K' be a prime field. Since G satisfies $(*)$ and $|K'| \leq |G|$, we have already seen that $K'G$ is primitive. In view of Lemma 4.1, we need only show that $\Delta(G) = 1$.

Let g be a non-identity element in G. We can see that there exist infinite conjugate elements of g. In fact, if it is not true, then the set M of conjugate elements of g in G is a finite set. Since G satisfies $(*)$, for M, there exists $x_1, x_2 \in G$ such that M^{x_1} and M^{x_2} are mutually reduced. Since g is in M, $(x_1^{-1}gx_1)(x_2^{-1}fx_2)^{-1} \neq 1$ for any $f \in M$, and thus $x_1^{-1}gx_1 \neq x_2^{-1}fx_2$. Hence $(x_1x_2^{-1})^{-1}g(x_1x_2^{-1}) \neq f$ for all $f \in M$, which implies a contradiction $x^{-1}gx \notin M$, where $x = x_1x_2^{-1}$. This completes the proof of theorem. $\square$

We call the free product $A * B$ of two non-identity groups A and B a strict free product provided that it is not isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. In addition, we define a group G to be a locally strict free product if for each finite number of elements $g_1, \cdots, g_m$ in G, there exists a subgroup H of G which is isomorphic to a strict free product such that $\{g_1, \cdots, g_m\} \subset H$. The following corollary, which generalizes the result of [6], follows from Theorem 2.1:

Corollary 4.4. *Let R be a domain and G a locally strict free product. Suppose that G has a free subgroup whose cardinality is the same as that of G. If $|R| \leq |G|$ then the group ring RG is primitive.*

In particular, KG is primitive for any field K.

5 Proof of Theorem 2.2

Throughout this section, $F = \langle X \rangle$ denotes the free group with a base X. Let $G = \langle X \mid R \rangle$ denote the one relator group with the set of generators X with a relation R, where R is a cyclically reduced word in F. For a word W in F, if $R = W^n$, $n > 1$ and W is not a proper power in F, then G is called a one relator group with torsion. Let W be a word in F. We denote the normal closure of W in F by $\mathcal{N}_F(W)$. For a cyclically reduced word W, $\mathcal{W}_F(W)$ denotes the set of all cyclically reduced conjugates of both W and W^{-1}. If $W_i, \cdots, W_t$ are reduced words in F and $W = W_i \cdots W_t$ is also reduced, that is, there is no cancellation in forming the product $W_i \cdots W_t$, then we write $W \equiv W_i \cdots W_t$. For $Y \subset X$, $\langle Y \rangle_G$ is the subgroup of G generated by the homomorphic image in G of Y.

Lemma 5.1. *Let $n > 1$, and let $G = \langle X \mid R \rangle$, where W be a cyclically reduced word in F and $R = W^n$.*

(1) (See [18, Theorem], cf. [8]) *If $1 \neq V \in \mathcal{N}_F(R)$, then V contains a subword $S^{n-1}S_0$, where $S \equiv S_0S_1 \in \mathcal{W}_F(W)$ and every generator which appears in W appears in S_0.*

(2) (See [12, Theorem]) *The centralizer of every non-trivial element in G is a cyclic group.*

Lemma 5.2. *For $n > 1$, let $G = \langle X \mid R \rangle$ with $|X| > 1$, where $R = W^n$ and W is a cyclically reduced word in F.*

(1) If $S, T \subseteq X$, then $\langle S \rangle_G \cap \langle T \rangle_G = \langle S \cap T \rangle_G$.

(2) $\Delta(G) = 1$.

Proof. (1): If $S \subseteq T$ or $T \subseteq S$, then the assertion is clear, and so we may assume $S \not\subseteq T$ and $T \not\subseteq S$. It is obvious that $\langle S \rangle_G \cap \langle T \rangle_G \supseteq \langle S \cap T \rangle_G$. Suppose, to the contrary, that $\langle S \rangle_G \cap \langle T \rangle_G \supsetneq \langle S \cap T \rangle_G$. Then there exist reduced words

$u = u(s, a, \cdots, b)$ in $\langle S\rangle \setminus \langle S \cap T\rangle$ and $v = v(t, c, \cdots, d)$ in $\langle T\rangle \setminus \langle S\cap T\rangle$ such that $uv \in \mathcal{N}_F(R)$, where $a, \cdots, b \in S$, $c, \cdots, d \in T$, $s \in S\setminus(S\cap T)$, and $t \in T\setminus(S\cap T)$. Let w be the reduced word for uv, say $w \equiv u_1v_1$, where $u \equiv u_1u_2$ and $v \equiv u_2^{-1}v_1$. Then $w \equiv u_1v_1 \in \mathcal{N}_F(R)$. However, u_1 involves s but not t, and v_1 involves t but not s, which contradicts the assertion of Lemma 5.1 (1).

(2): Suppose , to the contrary, $\Delta(G) \neq 1$; thus there exists $1 \neq g \in G$ such that $[G : C_G(g)] < \infty$. By Lemma 5.1 (2), $C_G(g)$ is cyclic and in fact infinite cyclic because $|G|$ is not finite. Thus G is virtually cyclic and so, as is well-known, there exists a normal subgroup N of finite order such that G/N is isomorphic to either the infinite cyclic group $\mathbb{Z}$ or the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ (See [9, 137p]).

Since a one relator group with torsion is isomorphic to neither $\mathbb{Z}$ nor $\mathbb{Z}_2 * \mathbb{Z}_2$, we may assume $N \neq 1$. In both cases of $G/N \simeq \mathbb{Z}$ and $G/N \simeq \mathbb{Z}_2 * \mathbb{Z}_2$, there exists $x \in G \setminus N$ such that $\langle x\rangle_G$ is a infinite cyclic subgroup of G. Since $|N|$ is finite, then it is easily seen that there exists $m > 0$ such that $x^{-m}fx^m = f$ for all $f \in N$, which implies $N \subset C_G(x^m)$; a contradiction, because a infinite cyclic group does not contain non-trivial finite subgroups. $\square$

Let $X = \{x_1, x_2, \cdots, x_m\}$ with $m > 1$ and $F = \langle X\rangle$. To avoid unnecessary subscripts, we denote generators, $x_1, x_2, \cdots, x_m$, by $t, a, \cdots, b$. We consider the one relator group $G = \langle X\ |R\rangle$, where $R = W^n$, $n > 1$ and $W = W(t, a, \cdots, b)$ is a cyclically reduced word which is not a proper power. We assume that all generators appear in W. We shall see that there exists a normal subgroup L of G such that G/L is cyclic and L satisfies the assumption in Corollary 4.4. That is, G has the following type of subgroup G_∞ and L is a subgroup of it:

$$G_\infty = \langle X_\infty \mid R_i,\ i \in \mathbb{Z}\rangle \text{ with } R_i = W_i^n (n > 1), \tag{7}$$

where $X_\infty = \{a_j, \cdots, b_j \mid j \in \mathbb{Z}\}$ and for each $i \in \mathbb{Z}$, W_i is a cyclically reduced word in the free group $F_\infty = \langle X_\infty\rangle$. Let α_*, $\cdots$, β_* be respectively the minimum subscripts on a, $\cdots$, b occurring in W_0, and let $\alpha^*, \cdots, \beta^*$ be the maximum subscript on a, $\cdots$, b occurring in W_0, respectively. That is,

$$W_i = W_i(a_{\alpha_*+i}, \cdots, a_{\alpha^*+i}, \cdots, b_{\beta_*+i}, \cdots, b_{\beta^*+i}).$$

Let μ be the maximum number in $\{\alpha^* - \alpha_*, \cdots, \beta^* - \beta_*\}$. For $t \in \mathbb{Z}$, we set subgroups Q_t and P_t of G_∞ as follows:

$$\begin{cases} \text{For } \mu \neq 0, \\ Q_t \ = \langle a_{t+i}, \cdots, b_{t+j} \mid \alpha_* \leq i \leq \alpha^*,\ \cdots,\ \beta_* \leq j \leq \beta^*\rangle_{G_\infty}, \\ P_t \ = \langle a_{t+i}, \cdots, b_{t+j} \mid \alpha_* \leq i \leq \alpha^* - 1,\ \cdots,\ \beta_* \leq j \leq \beta^* - 1\rangle_{G_\infty}. \\ \text{For } \mu = 0, \\ Q_t \ = \langle a_{t+\alpha_*},\ \cdots,\ b_{t+\beta_*}\rangle_{G_\infty}, \\ P_t \ = 1. \end{cases} \tag{8}$$

Then P_t is a subgroup of Q_t and Q_t has the following presentation:

$$Q_t \simeq \langle a_{t+\alpha_*}, \cdots, a_{t+\alpha^*}, \ \cdots, \ b_{t+\beta_*}, \cdots, b_{t+\beta^*} \mid R_t \rangle. \tag{9}$$

In what follows, let $\nu = \beta^* - \beta_*$, and replacing the order of $a_i, \cdots, b_i$ in X_∞ if necessary, we may assume that $\mu = \alpha^* - \alpha_* \geq \cdots \geq \beta^* - \beta_* = \nu$. In view of the Magnus' method for Freiheitssatz, we may identify G_∞ as the union of the chain of the following G_i's (see [11] or [10]):

$$\begin{array}{l} G_\infty = \bigcup_{i=0}^{\infty} G_i, \text{ where} \\ G_0 = Q_0, \quad G_{2i} = Q_{-i} *_{P_{-i+1}} G_{2i-1}, \text{ and } \ G_{2i+1} = G_{2i} *_{P_{i+1}} Q_{i+1}. \end{array} \tag{10}$$

By lemma 5.2 (1), we can get the next lemma:

Lemma 5.3. *If H is a subgroup of G_∞ generated by a finite subset Y of X_∞; namely $H = \langle Y \rangle_{G_\infty}$, then there exists a positive integer t such that $H \subseteq G_{2(t-1)}$ and $H \cap P_t = 1$.*

Lemma 5.4. *If G_∞ and W_i are as in (7), then for each finite number of elements $g_1, \cdots, g_m$ in G_∞, there exists an integer t such that $\langle g_1, \cdots, g_m, W_t \rangle_{G_\infty}$ is the free product $\langle g_1, \cdots, g_m \rangle_{G_\infty} * \langle W_t \rangle_{G_\infty}$.*

Proof. Let Y be the subset of X_∞ consisting of generators appeared in g_i for all $i \in \{1, \cdots, m\}$. By virtue of Lemma 5.3, for $H = \langle Y \rangle_{G_\infty}$, there exists $t > 0$ such that $H \subseteq G_{2(t-1)}$ and $H \cap P_t = 1$.

Now, by (10), $G_{2t-1} = G_{2(t-1)} *_{P_t} Q_t$, where Q_t is as described in (9) and P_t is as described in (8). Since $W_t^n = R_t$ is the relator of Q_t, we have $\langle W_t \rangle_{G_\infty} \subset Q_t$. As is well known, $W_t^m \neq 1$ in Q_t for $1 \leq m < n$. Moreover, it holds that $P_t \cap \langle W_t \rangle_{Q_t} = 1$. In fact, if not so, there exists $m > 0$ such that $W_t^m \in P_t$ in Q_t. Since P_t is a free subgroup of Q_t by Freiheitssatz, we have that $1 \neq (W_t^m)^n = (W_t^n)^m$ in Q_t. However, this contradicts the fact that W_t^n is the relator of Q_t. We have thus shown that $P_t \cap \langle W_t \rangle_{Q_t} = 1$. Combining this with $H \cap P_t = 1$, we see that $\langle Y, W_t \rangle_{G_{2t-1}} = \langle Y \rangle_{G_{2t-1}} * \langle W_t \rangle_{G_{2t-1}} = H * \langle W_t \rangle_{G_\infty}$. Since $\langle g_1, \cdots, g_m \rangle_{G_\infty} \subseteq H$, we have that $\langle g_1, \cdots, g_m, W_t \rangle_{G_\infty} = \langle g_1, \cdots, g_m \rangle_{G_\infty} * \langle W_t \rangle_{G_\infty}$. $\square$

Proof of Theorem 2.2 Let $G = \langle X \mid R \rangle$ be the one relator group with torsion, where $|X| > 1$, $R = W^n$, $n > 1$ and W is a cyclically reduced word which is not a proper power. If there exists $x \in X$ such that W contains none of x or x^{-1}, then G is a non-trivial free product of groups both of which are not isomorphic to $\mathbb{Z}_2$. Hence we may assume that $X = \{x_1, \cdots, x_m\}$ $(m > 1)$ and W contains either x_i or x_i^{-1} for all $i \in \{1, \cdots, m\}$. In this case, the cardinality of G is countable, and it is well-known that G has a non-cyclic free subgroup. Moreover, by Lemma 5.2 (2), we see that $\Delta(G) = 1$, and therefore, combining Corollary 4.4 with [19, Theorem 1], it suffices to show that there exists a normal subgroup L of G such that G/L is cyclic and L satisfies the following condition (C):

(C) For any $g_1, \cdots, g_l \in L$, there exists a free product $A * B$ in the set of subgroups of L such that $B \neq 1, a^2 \neq 1$ for some $a \in A$, and $g_1, \cdots, g_l \in A * B$.

There are now two cases to consider: whether or not the exponent sum $\sigma_x(W)$ of W on some generator x is zero.

If for each $x \in X$, $\sigma_x(W) \neq 0$, say $\sigma_{x_1}(W) = \alpha$ and $\sigma_{x_2}(W) = \beta$, then by the Magnus' method for Freiheitssatz, $G \simeq \langle a^\beta, x_2, \cdots, x_m \mid R^* \rangle \subset E$, where $R^* = (W^*)^n$, $W^* = W^*(a^\beta, x_2, \cdots, x_m)$ and $E = \langle a, x_2, \cdots, x_m \mid R^* \rangle$. Let $N = \mathcal{N}_{F_*}(x_2 a^\alpha, x_3 \cdots, x_m)$, where $F_* = \langle a, x_2, \cdots, x_m \rangle$. Then we have that $N \supset \mathcal{N}_{F_*}(R^*)$ and $N/\mathcal{N}_{F_*}(R^*) \simeq G_\infty$, where G_∞ is as in (7), and so we may let $G_\infty = N/\mathcal{N}_{F_*}(R^*)$.

Let $F_G = \langle a^\beta, x_2, \cdots, x_m \rangle$ and $L = (N \cap F_G)/\mathcal{N}_{F_G}(R^*)$. Then we can easily see that L can be isomorphically embedded in G_∞ and that G is a cyclic extension of L.

Let $g_1, \cdots, g_l$ $(l > 0)$ be in L with $g_i \neq 1$. In case of $n > 2$, since $L \subset G_\infty$, by Lemma 5.4, there exists $t > 0$ such that $\langle g_1, \cdots, g_l \rangle_{G_\infty} * \langle W_t^* \rangle_{G_\infty}$. We have then that $1 \neq W_t^* \in L$ and $(W_t^*)^2 \neq 0$ because $n > 2$, and so L satisfies the condition (C). On the other hand, in case of $n = 2$, let $p > 0$ be the maximum number such that either $a^{p\beta}$ or $a^{-p\beta}$ is appeared in $W^* = W^*(a^\beta, x_2, \cdots, x_m)$. Set $v = a^{(p+1)\beta} x_2 a^{-(p+1)\beta} x_2^{-1}$ so that $v \in F_G$. Moreover, since $\sigma_a(v) = 0$ and $\sigma_{x_2}(v) = 0$, the homomorphic image $\overline{v}$ of v is contained in L. Suppose that $\overline{v}^2 = 1$; namely, $v^2 \in \mathcal{N}_{F_G}(R^*)$. In view of Lemma 5.2 (1), a reduced word v^2 contains a subword $S_0 S_1 S_0$ such that $S_0 S_1$ is a cyclic shift of W^* and S_0 contains all generators appeared in W^*. Since only two letters a and x_2 are appeared in v^2, we have that $W^* = W^*(a^\beta, x_2)$. Moreover, $S_0 S_1 S_0$ involves a subword of type $x_2^{\varepsilon_1} a^q x_2^{\varepsilon_2}$ with $|q| \leq |p\beta|$, where $\varepsilon_i = \pm 1$. However, since $|(p+1)\beta| > |q|$, there exists no such subword in v^2, which implies a contradiction. We have thus shown that $\overline{v}^2 \neq 1$. By virtue of Lemma 5.4, for $g_1, \cdots, g_l$ and $\overline{v}$, there exists $t > 0$ such that $\langle \overline{v}, g_1, \cdots, g_l \rangle_{G_\infty} * \langle W_t^* \rangle_{G_\infty}$. Since $1 \neq W_t^* \in L$ and $\overline{v}^2 \neq 1$, we have thus proved that L satisfies the condition (C).

If W has a zero exponent sum $\sigma_x(W)$ on x for some $x \in X$, say $\sigma_{x_1}(W) = 0$, then we set $N = \mathcal{N}_F(x_2, x_3 \cdots, x_m)$ and $L = N/\mathcal{N}_F(R)$, where $F = \langle x_1, x_2, \cdots, x_m \rangle$, $R = W^n$ and $W = W(x_1, \cdots, x_m)$. It is obvious that $L \simeq G_\infty$ and G is a cyclic extension of L. Moreover, we can easily see that L satisfies the condition (C). This completes the proof of the theorem.

References

[1] B. O. Balogun, *On the primitivity of group rings of amalgamated free products*, Proc. Amer. Math. Soc., 106(1)(1989), 43-47

[2] M. A. Chaudhry, M. J. Crabb and M. McGregor, *The primitivity of semigroup algebras of free products* Semigroup Forum, **54**(2)(1997), 221-229

[3] O. I. Domanov, *A prime but not primitive regular ring*, Uspekhi. Mat. Nauk., **32)**(1977), 219-220

[4] O. I. Domanov, *Primitive group algebras of polycyclic groups*, Sibirsk. Mat. Ž., **19(1)**(1978), 37-43

[5] D. R. Farkas and D. S. Passman, *Primitive Noetherian group rings*, Comm. Algebra, **6(3)**(1978), 301-315.

[6] E. Formanek, *Group rings of free products are primitive*, J. Algebra, **26**(1973), 508-511

[7] E. Formanek and R. L. Snider, *Primitive group rings*, Proc. Amer. Math. Soc., **36**(1972), 357-360

[8] J. Howie and S. J. Pride, *A spelling theorem for staggered generalized 2-complexes, with applications*, Invent. math., 76 (1984), 55-74

[9] D. Juan-Pineda and Ian J. Leary, *On classifying spaces for the family of virtually cyclic subgroups*, Recent developments in algebraic topology, Contemp. Math., 407, Amer. Math. Soc., Providence, RI, 407(2006), 135-145

[10] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Ergeb. der Math. und ihrer Grenzgeb., 89, Springer, Berlin 1977

[11] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Wiley 1966, Second Edition, Dover Publication, New York, 1976

[12] B. B. Newman, *The soluble subgroups of a one-relator group with torsion*, J. Austral. Math. Soc., 16(1973), 278-285

[13] T. Nishinaka, *Group rings of proper ascending HNN extensions of countably infinite free groups are primitive*, J. Algebra, **317**(2007), 581-592

[14] T. Nishinaka, *Group rings of countable non-abelian locally free groups are primitive*, Int. J. algebra and computation, **21**(3) (2011), 409-431

[15] A. Ju. Ol'shanskiĭ, *An infinite simple torsion-free Noetherian group*, Izv. Akad. Nauk BSSR, Ser. Mat., 43(1979), 1328-1393.

[16] D. S. Passman, *Primitive group rings*, Pac. J. Math., **47**(1973), 499-506.

[17] J. E. Roseblade, *Prime ideals in group rings of polycyclic groups*, Proc. London Math. Soc., **36(3)**(1978), 385-447. Corrigenda "Prime ideals in group rings of polycyclic groups" Proc. London Math. Soc., **36(3)**(1979), 216-218.

[18] Paul E. Schupp, *A Strengthened Freiheitssatz*, Math. Ann., 221(1976), 73-80.

[19] A. E. Zalesskii, *The group algebras of solvable groups*, Izv. Akad. Nauk BSSR, ser Fiz. Mat., (1970), 13-21.

Semigroup rings over semiprime ring semigroups

Hisaya Tsutsui

Department of Mathematics
Embry-Riddle Aeronautical University
Prescott, AZ USA
E-mail address: Hisaya.Tsutsui@erau.edu

1 Introduction

The talk presented was based on an on-going joint work with Y.Hirano (Naruto) and B.Solie (ERAU). Our work is still it's initial stage. Corabolation with interested readers are highly desiable.

Let R be a ring, and let S be a semigroup. The semigroup ring $R[S]$ simultaneously encodes the semigroup structure of S and the ring structure resulting in an object of great utility in various areas of both ring theory and group theory. The interplay between the structures of R, S, and $R[S]$ is a topic with a long mathematical history full of fascinating results. For instance, much is already known about the structure of $R[G]$, where G is not just a semigroup but a group. Here, we find that the structure of $R[G]$ is exactly characterized by the structure of R and various finiteness properties of G. Maschke's famous theorem states that if G is a finite group and K is a field whose characteristic does not divide $|G|$, then $K[G]$ is semisimple [3]. Further variations include the result that $R[G]$ is prime if and only if R is a prime ring and G has no nontrivial finite normal subgroup [1]. More generally, $R[G]$ is semiprime if and only if R is semiprime and the order of every normal finite subgroup of G is a non-zero-divisor in R [5].

Consequently, one may ask whether the primeness or semiprimeness of $R[S]$ can be characterized in the case where S is an arbitrary semigroup. This is a much more difficult question on which some progress has been made in recent years. For highly structured classes of semigroups, it is again possible to determine the structure of $R[S]$. For example, when S is a cancellative

---

This work is supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University, Kyoto Japan.

10.5281/zenodo.2593809

semigroup, then $R[S]$ is semiprime whenever R is semiprime [4]. When F is a field and S is finite, a version of Maschke's theorem shows that the semisimplicity of $R[S]$ is characterized by the nonsingularity of the structure matrix for S when viewed as a matrix over F [2].

We consider semigroup rings over a particular class of semigroups: those semigroups which arise as the multiplicative semigroup of a ring.

2 Some Results and Examples

Let K be a field and let S be a semigroup. Recall that the *semigroup ring* $K[S]$ consists of the set of all sums $\sum_{s\in S} k_s \hat{s}$ with $k_s = 0$ for all but finitely many $s \in S$. We equip $K[S]$ with the usual addition and multiplication, where $\hat{s}\hat{t} = \widehat{st}$ for all $s, t \in S$.

Given a ring $(R,+,\cdot)$, we may forget addition and thereby obtain a semigroup $(R,\cdot)$ having both 0 and 1. We denote by $K[R]$ the semigroup ring $K[(R,\cdot)]$. Note that as $K[R] = K[0] \oplus Ann(K[0])$, $K[R]$ *is* not a prime ring. It is easy to observe that $K[R]$ is a direct sum of two simple rings if and only if every proper nonzero ideal of $K[R]$ is prime. Every prime ideal of $K[R]$ except $Ann(K[0])$ contains $K[0]$, and every ideal of $K[R]$ contained in $Ann(K[0])$ does not contain $K[0]$.

Theorem 1 *Let R be a ring and K be a field of characteristic zero. If $K[R]$ is semiprime, then R is semiprime. The converse holds if R is a commutative ring or a domain.*

Theorem 2 *Let R be a ring and let K be a field of characteristic zero. Then $K[R]$ is a semisimple Artinian ring if and only if R is a finite semisimple ring.*

For a ring R, we shall denote the Jacobson radical of R by $J(R)$.

Theorem 3 *Let R be a local ring with finitely many units, let R^* denote the group of units in R and let K be a field of characteristic zero. Then $J(K[R]) = \sum_{r\in J(R)} K(\hat{r} - \hat{0})$ and*
$K[R]/J(K[R]) \cong K[0] \oplus K[R^*]$.

Let K be an algebraically closed field of characteristic zero, and let $\mathbb{Z}_n$ denote the ring of integers modulo n. It is immediate that $\mathbb{Z}_n$ is semiprime if and only if n is squarefree, and thus we have the following proposition as a corollary of Theorem 1.

Proposition *Let K be an algebraically closed field of characteristic zero, and let $\mathbb{Z}_n$ denote the ring of integers modulo n. Then $K[\mathbb{Z}_n]$ is semiprime if and only if n is squarefree.*

We now present a few examples of the structure of $K[R]$ for some finite ring R.

Example 1 *It is clear that* $K[\mathbb{Z}_2] \cong K \oplus K$.

Let V be a vector space over K. We define a multiplication on the K-linear space $K \oplus V$ by the formula $(a,v)\cdot(b,w) = (ab, aw+bv)$ for any $a, b \in K,\ v, w \in V$. Then $K \oplus V$ becomes a K-algebra, which we denote by $K \ltimes V$.

Example 2 *Consider the ring $K[\mathbb{Z}_4]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1,2,3$. Then $K[\mathbb{Z}_4]$ is the direct sum of two sided ideals $K[0]$ and $S = Kg_1 + Kg_2 + Kg_3$. The identity of the ring S is g_1. Let us set $e_1 = \frac{1}{2}(g_1 - g_3)$, $e_2 = \frac{1}{2}(g_1 + g_3)$. Then e_1, e_2 are orthogonal central primitive idempotents of S and $g_1 = e_1 + e_2$.* We can easily see that $K[\mathbb{Z}_4] \cong K^2 \oplus (K \ltimes K)$.

Example 3 *Consider the ring $K[\mathbb{Z}_6]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1,2,\cdots,5$. Then $K[\mathbb{Z}_6]$ is the direct sum of $K[0]$ and $S = Kg_1 + Kg_2 + \cdots + Kg_5$. Set $e_1 = \frac{1}{2}(g_1 + g_5)$ and $e_2 = \frac{1}{2}(g_1 - g_5)$. We again have orthogonal central primitive idempotents e_1 and e_2 in S and $g_1 = e_1 + e_2$, and moreover $e_1 S \cong K^3$ and $e_2 S \cong K^2$. Thus we have that $K[\mathbb{Z}_6] \cong K^6$.*

Example 4 *Consider the ring $K[\mathbb{Z}_8]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1,2,\cdots,7$. Then $K[\mathbb{Z}_8]$ is the direct sum of two sided ideals $K[\hat{0}]$ and $S = Kg_1 + Kg_2 + \cdots + Kg_7$. The identity of the ring S is g_1. Let us set*

$$e_1 = \frac{1}{4}(g_1 - g_3 - g_5 + g_7)$$
$$e_2 = \frac{1}{4}(g_1 + g_3 - g_5 - g_7)$$
$$e_3 = \frac{1}{4}(g_1 - g_3 + g_5 - g_7)$$
$$e_4 = \frac{1}{4}(g_1 + g_3 + g_5 + g_7).$$

Then e_1, e_2, e_3, e_4 are orthogonal central primitive idempotents of S and $g_1 = e_1 + e_2 + e_3 + e_4$. We can easily see $e_1 S \cong K$, $e_2 S \cong K$, $e_3 S \cong K \ltimes K$, and $e_4 S \cong K \ltimes (K \oplus K)$. Therefore we have that $K[\mathbb{Z}_8] \cong K^3 \oplus (K \ltimes K) \oplus (K \ltimes (K \oplus K))$.

Example 5 *Consider the ring* $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}$ *of order* $|R| = 2^4 = 16$.

Then $R^* = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a \neq 0, b, c, d \in GF(2) \right\}$. We can easily see that R^* is the dihedral group D_8 of order 8 and so $K[R^*] \cong K^4 \oplus M_2(K)$. Therefore, by Theorem 3, we have that $K[R]/J(K[R]) \cong K^5 \oplus M_2(K)$.

Example 6 *Let* $M_2(GF(2))$ *denote the ring of* 2×2 *matrices over the field* $GF(2)$. *Then we can prove that* $Z[M_2(GF(2))]$ *is a semiprime ring. Let us set* $H = M_2(GF(2)) - GL_2(GF(2))$. *Then we can see that* $Q[H] \cong Q \oplus M_3(Q)$. *In fact, let*

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad e_7 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_8 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad e_9 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then the elements $E = \hat{O},\ F_1 = \hat{e}_1 - \hat{O},\ F_2 = \hat{e}_2 - \hat{O},$

$$F_3 = (\hat{e}_1 - \hat{O}) + (\hat{e}_2 - \hat{O}) + (\hat{e}_3 - \hat{O}) + (\hat{e}_4 - \hat{O}) - (\hat{e}_5 - \hat{O})$$
$$- (\hat{e}_6 - \hat{O}) - (\hat{e}_7 - \hat{O}) - (\hat{e}_8 - \hat{O}) + (\hat{e}_9 - \hat{O})$$

are primitive orthogonal idempotents, and $Q[H] = QE \oplus Q[H](F_1 + F_2 + F_3) \cong Q \oplus M_3(Q)$.
Since $GL_2(GF(2)) \cong S_3$, we have $Q[M_2(GF(2))]/Q[H] \cong Q[S_3]$.
It is easily see that $Q[S_3] \cong Q \oplus Q \oplus M_2(Q)$. Hence $Q[M_2(GF(2))]$ is isomorphic to the semisimple Artinian ring $Q \oplus Q \oplus Q \oplus M_2(Q) \oplus M_3(Q)$.

References

[1] Ian G. Connell. *On the group ring*. Canad. J. Math., 15:650-685, 1963.

[2] Michael P. Drazin. *Maschke's theorem for semigroups*. J. Algebra,72(10): 269-278, 1981.

[3] T. Y. Lam. *A frst course in noncommutative rings*, volume 131 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 2001.

[4] Jan Okninski. *Prime ideals of cancellative semigroups*. Comm. Algebra, 32(7):2733-2742, 2004.

[5] Donald S. Passman. *Infnite group rings*. Marcel Dekker, Inc., New York, 1971. Pure and Applied Mathematics, 6.

Separable polynomials in skew polynomials rings

Satoshi Yamanaka

Department of Integrated Science and Technology
National Institute of Technology, Tsuyama College
yamanaka@tsuyama.kosen-ac.jp

Abstract

Separable polynomials in skew polynomial rings have already been studied by K. Kishimoto, T. Nagahara, Y. Miyashita, S. Ikehata, and G. Szeto. In particular, Nagaraha gave the necessary and sufficient condition for Galois polynomial of degree 2 in the skew polynomial ring of derivation type. In this paper, we shall introduce some fundamental results of separable polynomials in skew polynomial rings which were already known. Further, we shall give a generalization of the Nagahara's result for Galois polynomials of degree 2 to general prime degree p.

Mathematics Subject Classification : 16S36, 16W25

Keywords: separable extension, separable polynomial, Hirata separable extension, Hirata separable polynomial, G-Galois extension, Galois polynomial, skew polynomial ring, derivation

1 Introduction

My talk at the conference was based on the paper [30] which is a joint work with S. Ikehata. The contents of this paper therefore overlaps with the publication.

Throughout this paper, every ring has identity 1, its subring contains 1, and A/B will represent a ring extension, that is, B is a subring of A. We say that A/B is a *separable extension* if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \mapsto ab$ splits, and A/B is a *Hirata separable extension* if $A \otimes_B A$ is A-A-isomorphic to a direct summand of a finite direct sum of copies of A. Moreover, A/B is called a *G-Galois extension* with a finite group G of automorphisms of A if $B = A^G$ (the fixed ring of G in A) and there are some finite number of elements $x_i,\ y_i \in A$ such that $\sum_i x_i\sigma(y_i) = \delta_{1,\sigma}$ for any $\sigma \in G$. Then we call $\{x_i,\ y_i\}$ a G-Galois coordinate system for A/B. It is well known that both a Hirata separable extension and a G-Galois extension are separable. The notion of Hirata separable extensions was introduced by K. Hirata as a generalization of Azumaya algebras. So far we called this extension as "H-separable extension". However, recently G. Szeto and L. Xue have begun to call it "Hirata separable extension". So we follow them.

10.5281/zenodo.2579479

Now, let B be a ring, ρ an automorphism of B, D a ρ-derivation (i.e. D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\rho(\beta) + \alpha D(\beta)$ for any $\alpha, \beta \in B$). $B[X;\rho,D]$ will mean the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. We write $B[X;\rho] = B[X;\rho,0]$ and $B[X;D] = B[X;1,D]$. By $B[X;\rho,D]_{(0)}$ we denote the set of all monic polynomials g in $B[X;\rho,D]$ such that $gB[X;\rho,D] = B[X;\rho,D]g$. For $f \in B[X;\rho,D]_{(0)}$, the residue ring $B[X;\rho,D]/fB[X;\rho,D]$ is a free ring extension of B. We say that $f \in B[X;\rho,D]_{(0)}$ is a *separable* (resp. *Hirata separable*, *Galois*) *polynomial* in $B[X;\rho,D]$ if $B[X;\rho,D]/fB[X;\rho,D]$ is a separable (resp. Hirata separable, G-Galois) extension of B. These provide typical and essential examples of separable extensions, Hirata separable extensions, and G-Galois extensions. K. Kishimoto, T. Nagahara, Y. Miyashita, S. Ikehata, and G. Szeto studied extensively separable polynomials in skew polynomial rings (See References). In particular, Nagahara thoroughly studied separable polynomials of degree 2 in skew polynomial rings (cf. [18], [19], [20], [22], [23]). As one of his results, he gave the necessary and sufficient condition for Galois polynomial of degree 2 in $B[X;D]$. The pourpose of this paper is to give a generalization of Nagahara's result for polynomials of degree 2 to general prime degree p.

In section 2, we shall introduce some fundamental results of separable polynomials in skew polynomial rings which was already known. We shall study the relationship between a Hirata separable extension and a G-Galois extension (or a purely inseparable extension of exponent one).

In section 3, we shall study Galois polynomials of prime degree p in $B[X;D]$. We shall give a necessary and sufficient condition for Galois polynomials of the form $X^p - Xa - b$.

In the following we shall use the following conventions:

Z = the center of B.

$U(Z)$ = the set of all invertible elements in Z.

u_r (resp. u_ℓ) = the right (resp. left) multiplication in B by $u \in B$.

$I_u = u_r - u_\ell$ (the inner derivation of B by $u \in B$).

$\rho|Z$ (resp. $D|Z$) = the restriction of ρ (resp. D) in Z.

$B^\rho = \{\alpha \in B \mid \rho(\alpha) = \alpha\}$ and $Z^\rho = Z \cap B^\rho$.

$B^D = \{\alpha \in B \mid D(\alpha) = 0\}$ and $Z^D = Z \cap B^D$.

2 Some results of separable polynomials

In this section, we shall introduce some fundamental results of separable polynomials in skew polynomial rings which were already known. First, we shall state some properties of the coefficients of skew polynomials which have been obtained in [5].

Lemma 2.1. ([5, Lemma 1.1]) *If* $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ *is in* $B[X;\rho,D]_{(0)}$, *then* $\alpha f = f\rho^m(\alpha)$ *for any* $\alpha \in B$ *and* $Xf = f(X - \rho(a_{m-1}) + a_{m-1})$, *and conversely.*

Lemma 2.2. ([5, Lemma 1.3]) *Let* $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ *be in* $B[X;\rho]$. *Then* f *is in* $B[X;\rho]_{(0)}$ *if and only if*
(1) $\alpha a_i = a_i\rho^{m-i}(\alpha) \quad (\alpha \in B,\ 0 \le i \le m-1)$.
(2) $\rho(a_{i-1}) - a_{i-1} = a_i(\rho(a_{m-1}) - a_{m-1}) \quad (1 \le i \le m-1)$.
(3) $a_0(\rho(a_{m-1}) - a_{m-1}) = 0$.

Lemma 2.3. ([5, Lemma 1.6]) *Let* $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ *be in* $B[X;D]$. *Then* f *is in* $B[X;D]_{(0)}$ *if and only if*
(1) $a_i\alpha = \displaystyle\sum_{j=0}^{i} \binom{j}{i} D^{j-i}(\alpha)a_j \quad (\alpha \in B,\ 0 \le i \le m-1)$.
(2) $D(a_i) = 0 \quad (0 \le i \le m-1)$.

In [17], Y. Miyashita gave the followings as characterizations of the separable and Hirata separable polynomials in skew polynomial rings.

Proposition 2.4. ([17, Theorem 1.8]) *Let* $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ *be in* $B[X;\rho,D]_{(0)}$, $A = B[X;\rho,D]/fB[X;\rho,D]$ *and* $x = X + fB[X;\rho,D]$. *Then* f *is separable in* $B[X;\rho,D]$ *if and only if there exists* $h \in A$ *such that* $\rho^{m-1}(\alpha)h = h\alpha$ *for any* $\alpha \in B$). *and* $\sum_{j=0}^{m-1} y_j h x^j = 1$, *where* $y_j = x^{m-j-1} + x^{m-j-2}a_{m-1} + \cdots + xa_{j+2} + a_{j+1}$ $(0 \le j \le m-2)$ *and* $y_{m-1} = 1$.

Proposition 2.5. ([17, Theorem 1.9], [6, Theorem 1.1]) *Let* $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ *be in* $B[X;\rho,D]_{(0)}$, *and* $A = B[X;\rho,D]/fB[X;\rho,D]$. *Then* f *is Hirata separable in* $B[X;\rho,D]$ *if and only if there exist* $g_i,\ h_i \in A$ *such that* $\alpha g_i = g_i\alpha$, $\rho^{m-1}(\alpha)h_i = h_i\alpha$ *for any* $\alpha \in B$, $\sum_i g_i x^{m-1} h_i = 1$, *and* $\sum_i g_i x^k h_i = 0$ $(0 \le k \le m-2)$.

Recently in [29], the author and S. Ikehata gave alternative proofs of Proposition 2.4 and Proposition 2.5 in $B[X;\rho]$ and $B[X;D]$, respectively. The followings are some part of Ikehata's results concerning Hirata sepaarable polynomials in skew polynomials rings.

Proposition 2.6. ([8, Lemma 1]) *If* $B[X;\rho]$ *contains a Hirata separable polynomial* f *of degree* $m \ge 2$, *then there holds the following:*
(1) f *is of the form* $X^m + a_0$, a_0 *is invertible in* B, *and* $\rho^m = (a_0^{-1})_\ell(a_0^{-1})_r$.
(2) *Every Hirata separable polynomial of degree* ≥ 2 *in* $B[X;\rho]$ *is of the form* $X^m + a_0c$, *where* $c \in U(Z) \cap B^\rho$.

Proposition 2.7. ([10, Theorem 2.2]) *If* $B[X;D]$ *contains a Hirata separable polynomial* f *of degree* $m \ge 2$, *then there holds the following:*

(1) *B is of prime characteristic p and f is a p-polynomial of the form*

$$\sum_{j=0}^{e} X^{p^j} b_{j+1} + b_0 \ (m = p^e), \ \ b_{j+1} \in Z^D \ (0 \le j \le e), \ \ \textit{and} \ \ b_0 \in B^D.$$

(2) *Every Hirata separable polynomial in $B[X;D]$ is of the form $f + c$, where $c \in Z^D$.*

Proposition 2.8. ([12, Theorem 3.3]) *Let B be of prime characteristic p, and $f = X^p - Xa - b$ in $B[X;D]_{(0)}$. If there exists an element z in Z such that $D(z)$ is invertible in Z, then f is a Hirata separable polynomial in $B[X;D]$. In addition, if z is an invertible element in Z, then f is a Galois polynomial in $B[X;D]$.*

Now we say that a ring extension A/B of commutative rings is a *purely inseparable extension of exponent one with δ* if ${}_BA$ is a finitely generated module of finite rank and $\mathrm{Hom}\,({}_BA, {}_BA) = B[\delta]$ (the subring generated by B_ℓ and δ), where δ is a derivation of A and $B = \{a \in A \mid \delta(a) = 0\}$. Purely inseparable ring extension of exponent one has been studied by S. Yuan and G. Georgantas (cf. [2], [31], [32]). In the theory, derivations play a role analogous to that of automorphisms in the theory of cyclic Galois extensions.

Hirata separable extensions have a close relationship with G-Galois extension (or purely inseparable extensions of exponent one). When the coefficient ring is commutative, the existence of Hirata separable polynomials in skew polynomial rings has been characterized in terms of G-Galois extensions (or purely inseparable extensions) and Azumaya algebra as follows:

Proposition 2.9. ([6, Theorem 2.2]) *Let B be a commutative ring. Then the following are equivalent:*

(1) *B/B^ρ is a G-Galois extension with G of order m.*

(2) *$B[X;\rho]_{(0)}$ contains a Hirata separable polynomial of degree m.*

(3) *$B[X;\rho]_{(0)}$ contains a polynomial f of degree m such that $B[X;\rho]/fB[X;\rho]$ is an Azumaya B^ρ-algebra.*

Proposition 2.10. ([6, Theorem 3.3]) *Let B be a commutative ring. Then the following are equivalent:*

(1) *${}_{B^D}B$ is a finitely generated projective module of rank m and* $\mathrm{Hom}\,({}_{B^D}B, {}_{B^D}B)$ $= B[D]$, *that is, B/B^D is a purely inseparable extension of exponent one with D.*

(2) *$B[X;D]_{(0)}$ contains a Hirata separable polynomial of degree m.*

(3) *$B[X;D]_{(0)}$ contains a polynomial f of degree m such that $B[X;D]/fB[X;D]$ is an Azumaya B^D-algebra.*

When the coefficient ring is noncommutative, it have been studied whether Proposition 2.9 and Proposition 2.10 are also true or not. Concerning this, G. Szeto and L. Xue gave an affirmative answer for Hirata separable polynomials in $B[X;\rho]$.

Proposition 2.11. ([28, Theorem 3.6]) *Let* $f = X^m - u$ $(m \geq 2)$ *is in* $B[X;\rho]_{(0)}$. *The following are equivalent.*

(1) f *is Hirata separable in* $B[X;\rho]$.

(2) u *is invertible in* B *and* Z/Z^ρ *is a* G*-Galois extension, where* G *is generated by* $\rho|Z$ *of order* m.

Proposition 2.11 shows that the existence of Hirata separable polynomials in $B[X;\rho]$ is characterized in terms of G-Galois extensions. Concerning the relationship between Hirata separable polynomials in $B[X;D]$ and purely inseparable extensions of exponent one, the following is already known.

Proposition 2.12. ([10, Proposition 2.3]) *Let* B *be of prime characteristic* p, *and* $\delta = D|Z$. *Assume that* Z/Z^δ *is a purely inseparable extension of exponent one with* δ, Z *is a projective module over* Z^δ *of rank* p^e, *and* δ *satisfies the minimal polynomial* $t^{p^e} + t^{p^{e-1}}\alpha_e + \cdots + t^p\alpha_2 + t\alpha_1$ $(\alpha_i \in Z^\delta)$. *If there exists an element* u *in* B^D *such that* $D^{p^e} + (\alpha_e)_\ell D^{p^{e-1}} + \cdots + (\alpha_2)_\ell D^p + (\alpha_1)_\ell D = I_u$, *then* $f = X^{p^e} + t^{X^{e-1}}\alpha_e + \cdots + X^p\alpha_2 + X\alpha_1 - u$ *is Hirata separable in* $B[X;D]$.

However, we do not prove yet the converse of Proposition 2.12. This is one of the open problems.

Open question 1. If $B[X;D]$ contains a Hirata separable polynomial, then is Z/Z^D a purely inseparable extension of exponent one?

As was shown in [13, Theorem 3.2], the above question is true if Z is a semiprime ring and $f = X^p - Xa - b$ is Hirata separable in $B[X;D]$. Moreover, it is also true when $B = B^D Z$.

3 Galois polynomials in $B[X;D]$

In this section, we shall study Galois polynomials of prime degree p in $B[X;D]$. From now on, let B be of prime characteristic p, $f = X^p - Xa - b \in B[X;D]_{(0)}$, $A = B[X;D]/fB[X;D]$, and $x = X + fB[X;D] \in A$. First, we shall state some basic results which were already known. The following is easily verified by a direct computation.

Lemma 3.1. ([5, Corollary 1.7]) *Let* $f = X^p - Xa - b$ *be in* $B[X;D]$. *Then* f *is in* $B[X;D]_{(0)}$ *if and only if* $a \in Z^D$, $b \in B^D$, *and* $D^p(\alpha) - D(\alpha)a = \alpha b - b\alpha$ *for any* $\alpha \in B$.

Concerning Galois polynomials, the following is well known.

Lemma 3.2. ([15, Theorem 1.1 and Corollary 1.1], [12, Lemma 2.3]) *Let $f = X^p - X - b$ be in $B[X; D]_{(0)}$. Then f is a Galois polynomial over B. More precisely, let $A = B[X; D]/fB[X; D]$ and $x = X + fB[X; D]$. Then A/B is a G-Galois extension, where $G =< \sigma >$, and $\sigma : A \to A$ defined by $\sigma(\sum_i x^i d_i) = \sum_i (x+1)^i d_i$ is a B-automorphism of A of order p.*

Proof. The proof is already known. However recently, we gave a new proof which gives a G-Galois coordinate system for A/B concretely. So we shall state it here. First, it is easy to see that $A^G = B$. We shall prove that

$$\{1,\, x, \cdots, x^i, \cdots, x^{p-1};\, 1 - x^{p-1},\, (p-1)x^{p-2}, \cdots, (-1)^{i-1}\binom{p-1}{i}x^{p-i-1}, \cdots, -1\}$$

is a G-Galois coordinate system for A/B. Since

$$\begin{aligned} 1 = k^{p-1} &= (-x + \sigma^k(x))^{p-1} \\ &= \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} x^i \sigma^k(x^{p-1-i}) \quad (1 \le k \le p-1) \quad \text{and} \\ 0 = (-x+x)^{p-1} &= \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} x^i x^{p-1-i}, \end{aligned}$$

we obtain

$$\begin{aligned} 1 = 1 - 0 = 1 - &\sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} x^i x^{p-1-i} \\ &= 1 \cdot (1 - x^{p-1}) + \sum_{i=1}^{p-1} x^i \left\{ (-1)^{i-1} \binom{p-1}{i} x^{p-1-i} \right\} \quad \text{and} \\ 0 = 1 - 1 = 1 - &\sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} x^i \sigma^k(x^{p-1-i}) \\ &= 1 \cdot \sigma^k(1 - x^{p-1}) + \sum_{i=1}^{p-1} x^i \sigma^k \left\{ (-1)^{i-1} \binom{p-1}{i} x^{p-1-i} \right\} \quad (1 \le k \le p-1). \end{aligned}$$

Thus

$$\{1,\, x, \cdots, x^i, \cdots, x^{p-1};\, 1 - x^{p-1},\, (p-1)x^{p-2}, \cdots, (-1)^{i-1}\binom{p-1}{i}x^{p-i-1}, \cdots, -1\}$$

is a G-Galois coordinate system for A/B. This completes the proof. $\square$

In general, if we consider a polynomial $f = X^p - Xa - b \in B[X;D]_{(0)}$ and $a \neq 1$, it is not easy to check whether f is a Galois polynomial or not. In [18], T. Nagahara considered the case the characteristic of B is 2 and $f = X^2 - Xa - b \in B[X;D]_{(0)}$. He proved the following.

Proposition 3.3. [18, Theorem 3.7] *Let B be of characteristic 2, and $f = X^2 - Xa - b$ in $B[X;D]_{(0)}$. Then f is a Galois polynomial in $B[X;D]$ if and only if there exists an element s in $U(Z)$ such that $D(s) + as = 1$.*

In [18], Nagahara showed that if $f = X^2 - Xa - b$ is a Galois polynomial in $B[X;D]$, that is, $B[X;D]/fB[X;D]$ is a G-Galois extension over B with some finite group G, then the order of G is necessarily 2, and $G =< \sigma_s >$, where $\sigma_s(x) = x + s^{-1}$, with an element $s \in U(Z)$ such that $D(s) + as = 1$. Now we shall prove the following which is a generalization of Nagahara's result (Proposition 3.3). This is the main theorem in this paper.

Theorem 3.4. ([30, Theorem 2.2]) *Let $f = X^p - Xa - b$ be in $B[X;D]_{(0)}$. If f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$ with an element $s \in U(Z)$, then $s^{-1}(sD)^{p-1}(s) + s^{p-1}a = 1$. Conversely, if there exists an element $s \in U(Z)$ such that $s^{-1}(sD)^{p-1}(s) + s^{p-1}a = 1$ then f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$.*

To prove the theorem we need the following lemma.

Lemma 3.5. ([30, Lemma 2.1])

$$D^{p-1}(s^{p-1}) = -s^{-1}(sD)^{p-1}(s) \quad \textit{for any element } s \textit{ in } U(Z).$$

Proof. We set here $W = sX + 1$. Then we have

$$\alpha W = W\alpha + sD(\alpha) \quad \text{for any } \alpha \in B.$$

So, we see that $B[X;D] = B[W;sD]$. Then we have

$$\begin{aligned}
(X + s^{-1})^p = (s^{-1}W)^p &= (s^{-1})^p W^p + (s^{-1} \cdot sD)^{p-1}(s^{-1})W \\
&= (s^{-1})^p W^p + D^{p-1}(s^{-1})W \\
&= (s^{-1})^p (sX + 1)^p + D^{p-1}(s^{-1})(sX + 1) \\
&= (s^{-1})^p \{(sX)^p + 1\} + D^{p-1}(s^{-1})sX + D^{p-1}(s^{-1}) \\
&= (s^{-1})^p \{s^p X^p + (sD)^{p-1}(s)X + 1\} + D^{p-1}(s^{-1})sX + D^{p-1}(s^{-1}) \\
&= X^p + \{(s^{-1})^p (sD)^{p-1}(s) + D^{p-1}(s^{-1})s\}X + (s^{-1})^p + D^{p-1}(s^{-1}).
\end{aligned}$$

On the other hand, it follows from [14, page 190, Exercises 8] that

$$(X + s^{-1})^p = X^p + (s^{-1})^p + D^{p-1}(s^{-1}).$$

Then we have $(s^{-1})^p (sD)^{p-1}(s) + D^{p-1}(s^{-1})s = 0$, and hence $D^{p-1}(s^{p-1}) = -s^{-1}(sD)^{p-1}(s)$. This completes the proof. □

Now we shall prove Theorem 3.4

The proof of Theorem 3.4. Assume that f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$ with an element $s \in U(Z)$. Then since $\sigma_s(x^p - xa - b) = 0$, we have

$$\begin{aligned}(x+s^{-1})^p - (x+s^{-1})a - b &= x^p + (s^{-1})^p + D^{p-1}(s^{-1}) - xa - s^{-1}a - b \\ &= (s^{-1})^p + D^{p-1}(s^{-1}) - s^{-1}a = 0.\end{aligned}$$

Then $1 + D^{p-1}(s^{p-1}) = s^{p-1}a$, and hence we have $s^{-1}(sD)^{p-1}(s) + s^{p-1}a = 1$ by Lemma 2.1.

Conversely, assume that $s^{-1}(sD)^{p-1}(s) + s^{p-1}a = 1$ for some element s in $U(Z)$. We put here $\Delta = sD$. Then by the well known Hochschild formula ([16, Theorem 25.5]) and Lemma 1.1, we have

$$\begin{aligned}\Delta^p = (sD)^p &= s^pD^p + (sD)^{p-1}(s)D \\ &= s^p(aD + I_b) + (sD)^{p-1}(s)D \\ &= \{s^{p-1}a + s^{-1}(sD)^{p-1}(s)\}sD + I_{s^pb} \\ &= \Delta + I_{s^pb}.\end{aligned}$$

We set $Y = sX$. Then we obtain

$$\alpha Y = Y\alpha + \Delta(\alpha) \quad \text{and} \quad \alpha Y^p = Y^p\alpha + \Delta^p(\alpha) \quad (\alpha \in B).$$

So, we see that $B[X;D] = B[Y;\Delta]$ and $Y^p - Y - s^pb = (sX)^p - sX - s^pb = s^pX^p + (sD)^{p-1}(s)X - sX - s^pb = s^p(X^p - aX - b) = s^pf$. It follows from Lemma 1.2 that $g = Y^p - Y - s^pb = s^pf$ is a Galois polynomial in $B[Y;\Delta]$ with a Galois group of order p. Noting $B[X;D] = B[Y;\Delta]$ and $fB[X;D] = B[X;D]f = gB[Y;\Delta] = B[Y;\Delta]g$, we see that f is also a Galois polynomial in $B[X;D]$. Recall that $A = B[X;D]/fB[X;D]$ and $x = X + fB[X;D] \in A$. Then by Lemma 1.2, it is easily seen that we can take a Galois group $G =< \sigma_s >$, where $\sigma_s : A \to A$ defined by $\sigma_s(\sum_i x^i d_i) = \sum_i (x + s^{-1})^i d_i$ is a B-automorphism of S of order p. This completes the proof. □

Remark 1. In Theorem 3.4, we can see that

$$\begin{aligned}\{1, sx, \cdots, (sx)^i, \cdots, (sx)^{p-1};\ 1 - (sx)^{p-1}, (p-1)(sx)^{p-2}, \cdots, \\ (-1)^{i-1}\binom{p-1}{i}(sx)^{p-1-i}, \cdots, -1\}\end{aligned}$$

is a G-Galois coordinate system for A/B by the similar computation in the proof of Lemma 3.2.

Remark 2. In [18], Nagahara proved that if $f = X^2 - Xa - b$ is a Galois polynomial in $B[X;D]$, then necessarily the order of the Galois group is 2.

However, in general case we do not prove yet that if $f = X^p - Xa - b$ is a Galois polynomial in $B[X; D]$, then the order of the Galois group is p. This is one of the open problem.

Open question 2. If $f = X^p - Xa - b$ is a Galois polynomial in $B[X; D]$, then is the order of the Galois group necessarily p ?

In virtue of Lemma 3.5, we obtain the following as a direct consequence of Theorem 3.4.

Corollary 3.6. ([30, Corollary 2.4]) *Let $f = X^p - Xa - b$ be in $B[X; D]_{(0)}$. If there exists an element $y \in Z$ such that $D^{p-1}(y) - ya = 1$ and $y = -s^{p-1}$ for some element $s \in U(Z)$, then f is a Galois polynomial in $B[X; D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$. Conversely, if f is a Galois polynomial in $B[X; D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$ with an element $s \in U(Z)$, then $D^{p-1}(y) - ya = 1$ and $y = -s^{p-1}$.*

Finally, we shall state the following which is an easy consequence of Corollary 3.6.

Corollary 3.7. ([30, Corollary 2.5]) *Let $f = X^p - Xa - b$ be in $B[X; D]_{(0)}$. If there exists an invertible element $u \in Z^D$ such that $u^{p-1} = a$, then f is a Galois polynomial in $B[X; D]$ with a Galois group $G =< \sigma_{u^{-1}} >$, where $\sigma_{u^{-1}}(x) = x + u$.*

References

[1] F. DeMeyer and F. Ingraham, *Separable algebra over a commutative ring*, Lecture Notes in Math., **181** (1971), Springer, Berlin.

[2] G. Georgantas, *Inseparable Galois cohomology*, J. Algebra, **38** (1976), 368–379.

[3] K. Hirata and K. Sugano, *On semisimple extensions and separable extensions over noncommutative rings*, J. Math. Soc. Japan, **18** (1966), 360–373.

[4] S. Ikehata, *A note on separable polynomials in skew polynomial rings of derivation type*, Math. J. Okayama Univ., **22** (1980), 59–60.

[5] S. Ikehata, *On separable polynomials and Frobenius polynomials in skew polynomial rings*, Math. J. Okayama Univ., **22** (1980), 115–129.

[6] S. Ikehata, *Azumaya algebras and skew polynomial rings*, Math, J. Okayama Univ., **23** (1981), 19–32.

[7] S. Ikehata, *On separable polynomials and Frobenius polynomials in skew polynomial rings. II*, Math. J. Okayama Univ., **25** (1983), 23–28.

[8] S. Ikehata, *Azumaya algebras and skew polynomial rings. II*, Math, J. Okayama Univ., **26** (1984), 49–57.

[9] S. Ikehata, *On H-separable polynomials of prime degree*, Math. J. Okayama Univ., **33** (1991), 21–26.

[10] S. Ikehata, *Purely inseparable ring extensions and H-separable polynomials*, Math. J. Okayama Univ., **40** (1998), 55–63.

[11] S. Ikehata, *Purely inseparable ring extensions and Azumaya algebras*, Math. J. Okayama Univ., **41** (1999), 63–69.

[12] S. Ikehata, *On H-separable and Galois polynomials of degree p in skew polynomial rings*, Int. Math. Forum, **3** (2008), no. 29–32, 1581–1586.

[13] S. Ikehata, *On separable and H-separable polynomials of degree p in skew polynomial rings*, Int. J. Pure Appl. Math., **51** (2009), no.1, 149–156.

[14] N. Jacobson, *Lectures in abstract algebra. Vol III: Theory of fields and Galois theory*, D. Van Nostrand Co., Inc. (1964).

[15] K. Kishimoto, *On abelian extensions of rings. I*, Math. J. Okayama Univ., **14** (1970), 159–174.

[16] H. Matsumura, *Commutative ring theory. Translated from the Japanese by M. Reid. Cambridge Studies in Advanced Mathematics, 8.*, Cambridge University Press, Cambridge (1986).

[17] Y. Miyashita, *On a skew polynomial ring*, J. Math. Soc. Japan, **31** (1979), no. 2, 317–330.

[18] T. Nagahara, *On separable polynomials of degree* 2 *in skew polynomial rings*, Math. J. Okayama Univ., **19** (1976), 65–95.

[19] T. Nagahara, *On separable polynomials of degree* 2 *in skew polynomial rings II*, Math. J. Okayama Univ., **21** (1979), 167–177.

[20] T. Nagahara, *On separable polynomials of degree* 2 *in skew polynomial rings III*, Math. J. Okayama Univ., **22** (1980), 61–64.

[21] T. Nagahara, *A note on separable polynomials in skew polynomial rings of atutomorphism type*, Math. J. Okayama Univ., **22** (1980), 73–76.

[22] T. Nagahara, *On separable polynomials of degree* 2 *in skew polynomial rings IV*, Math. J. Okayama Univ., **23** (1981), 171–179.

[23] T. Nagahara, *Some H-separable polynomials of degree* 2, Math. J. Okayama Univ., **26** (1984), 87–90.

[24] T. Nagahara, *A note on imbeddings of noncommutative separable extensions in Galois extensions*, Houston J. Math., **12** (1986), 411–417.

[25] H. Okamoto and S. Ikehata, *On H-separable polynomials of degree* 2, Math. J. Okayama Univ., **32** (1990), 53–59.

[26] K. Sugano, *Separable extensions and Frobenius extensions*, Osaka J. Math., **7** (1970), 29–40.

[27] K. Sugano, *Note on cyclic Galois extensions*, Proc. Japan Acad., **57** (1981), Ser. A, 60–63.

[28] G. Szeto and L. Xue, *On the Ikehata theorem for H-separable skew polynomial rings*, Math. J. Okayama Univ., **40** (1998), 27–32.

[29] S. Yamanaka and S. Ikehata, *An alternative proof of Miyasita's theorem in a skew polynomial rings*, Int. J. Algebra, **21** (2012), 1011–1023

[30] S. Yamanaka and S. Ikehata, *On Galois polynomials of degree p in skew polynomial rings of derivation type*, Southeast Bull. Math., **37** (2013), 625–634.

[31] S. Yuan, *On logarithmic derivatives*, Bull. Soc. Math. France, **96** (1968), 41–52.

[32] S. Yuan, *Inseparable Galois theory of exponent one*, Trans. Amer. Math. Soc., **149** (1970), 163–170.

Internal Bousfield Localizations

Renaud Gauthier *

September 20, 2018

Abstract

We develop the notion of left and right Bousfield localizations in proper, cellular symmetric monoidal model categories with cofibrant unit, using homotopy function complexes defined by internal Hom objects instead of Hom sets.

*rg.mathematics@gmail.com

10.5281/zenodo.2579469

1 Introduction

From [TV], recall that one can define a stack as a prestack in $\mathrm{sPr}(T)$, the model category of simplicial presheaves over a simplicial model category T, as an object $F \in \mathrm{sPr}(T)$ satisfying hyperdescent, meaning being local with respect to a certain class of hypercovers. To be precise, we consider hypercovers $F \to G$ in $s\mathrm{sPr}(T)$ (viewed as constant simplicial objects), maps such that for all $n \geq 0$:

$$F^{\mathbb{R}\Delta^n} \to F^{\mathbb{R}\partial\Delta^n} \times^h_{G^{\mathbb{R}\partial\Delta^n}} G^{\mathbb{R}\Delta^n}$$

is a covering. Those turn out to be π_*-equivalences as defined in [TV], maps $F \to G$ in $\mathrm{sPr}(T)$ such that for all $n > 0$, we have an isomorphism of sheaves $\pi_n(F) \to \pi_n(G)$, among other things. Such a map $F \to G$ would be a $\mathbb{S}$-colocal object in the language of [Hi]. Suppose we consider objects $\tilde{k_0}$ other than the sphere spectrum $\mathbb{S}$ in this definition of a local equivalence, cosimplicial resolutions of some k_0, object of some class K_0. Suppose further we consider functors not valued in Set_Δ, the category of simplicial sets, but in some category $\mathcal{D}$, which for the moment we can suppose is a proper, cellular model category. It would be natural then for the sake of localization to use internal Hom objects for the definition of homotopy function complexes, as opposed to using Hom sets. As a matter of fact, we will closely follow Hirschhorn's work ([Hi]) regarding Bousfield localizations and make the appropriate changes.

We start with a $\mathbb{U}$-small pseudo-model category $(\mathcal{C}, W)$ ([TV]), $\mathcal{D}_0$ a proper, cellular, symmetric monoidal model category, and we consider the functor category $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Prestacks in this setting are functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}_0$ mapping equivalences to equivalences; if $x \to y$ is in W, then $Fx \to Fy$ is an equivalence in $\mathcal{D}_0$. If $\mathcal{D}_0 = \mathrm{Set}_\Delta$, one can use the classical Yoneda lemma: $Fx \simeq \mathrm{Hom}(h_x, F)$, $h_x = \mathrm{Hom}_{\mathcal{C}}(-, x)$ to see equivalence preserving functors as local objects. For $\mathcal{D}_0$-enriched functors, we must consider enriched Yoneda: $Fx \simeq \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}(h_x(y), Fy)$. Then $Fx \xrightarrow{\simeq} Fy$ will follow from $\underline{\mathrm{Hom}}(Qh_x, RF) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(Qh_y, RF)$, hence the need for defining a notion of local object using internal Hom objects. This leads us to defining the internal left Bousfield localization $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}} = \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\,\wedge}$ of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ with respect to $\Gamma = \{h_y \to h_x \mid x \to y \in W^{\mathrm{op}}\}$.

Our philosophy at this point departs markedly from the standard philosophy of Homotopical Algebraic Geometry ([TV]) in that we have no topology on $\mathcal{C}$, and we limit ourselves to considering only one condition defining π_*-local equivalences, namely $\pi_n(F) \xrightarrow{\simeq} \pi_n(G)$ for all $n > 0$, which we internalize, using not spheres, but arbitrary objects k_0 of some class of objects K_0 of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$:

$$\underline{\mathrm{Hom}}(\tilde{k_0}, \hat{F}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{k_0}, \hat{G}) \qquad (1)$$

where using the sphere spectrum $\mathbb{S}$ for π_*-local equivalences is replaced by using a cosimplicial resolution $\tilde{k_0}$ of a single object k_0. Here $\hat{F}$ is a fibrant approximation to F. (1) would define what it means for $F \to G$ to be an internal K_0-colocal equivalence. From there we are naturally led to considering an internal right Bousfield localization $R_{\mathcal{K}_0} L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\,\wedge}$ with respect to $\mathcal{K}_0$, the class of K_0-colocal equivalences.

Our main reference for Bousfield localizations will be [Hi], and we will use [Ho] as a reference regarding monoidal model categories.

In Section 2 we present the main construction, giving a category of prestacks $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\,\wedge} = L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ from localizing $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ along a subset Γ, followed by further taking a right Bousfield localization along a class of colocal equivalences, all concepts being redefined using internal Homs. In Section 3 we present foundational results, such as the enriched Yoneda lemma, and localization using internal Homs for homotopy function complexes. In Section 4 we discuss cardinality arguments needed in the proof of our main result, the existence of a left Bousfield localization using internal Homs instead of Hom sets. In Section 5 we present technical results needed to prove the existence of such a localization, which is itself given in Section 6. In Section 7 we give those results needed to prove the existence of a right Bousfield localization using internal Homs, which is stated and proved in Section 8.

Relation to past work: it was pointed out to the author by J. Gutierrez and D. White, that the present work is quite close to past work on the subject. In particular they both referenced two papers the author was wholly unaware of, namely [B] and [GR]. As a matter of fact, Barwick's work is so close to the present one, the original thought of using the Hom from a Quillen adjunction of two variables to define Bousfield localizations must be credited to him. Presently we discuss localizations of symmetric monoidal model categories, and some work has been done on the subject, albeit in a classical sense, not using internal Homs. This work can be found in [W] and [WY] where such localizations are referred to as monoidal Bousfield localizations.

To come back to [W] and [GR], there are slight differences to be noted. Barwick works with tractable categories, which are combinatorial, hence cofibrantly generated. We work with cellular model categories, which are also cofibrantly generated. Barwick observes that he does not know of any left proper combinatorial model category that is not tractable. To be safe we will suppose they are different. On our part, we do not see how to relate cellular model categories with tractable model categories. Our objects of study are different. Additionally, Barwick works with $\mathcal{V}$-enriched categories $\mathcal{C}$, for $\mathcal{V}$ a symmetric monoidal model category, and the "internal" Homs he uses, derived from a Quillen adjunction of two variables due to this enrichment, are objects in $\mathcal{V}$. In particular he shows that for a small site $\mathcal{C}$, for $\mathcal{V}$ a tractable symmetric monoidal model category with cofibrant unit, $\mathcal{V}^{\mathcal{C}}$ is a $\mathcal{V}$-model category, so is enriched in $\mathcal{V}$, on which we define an injective local model structure as an enriched left Bousfield localization of $\mathcal{V}^{\mathcal{C}}$ with its injective model structure. What we do instead is show that $\mathcal{V}^{\mathcal{C}}$ is a symmetric monoidal model category with an internal Hom, and it is those internal Homs we use in the definition of our version of Bousfield localizations, and it is for this reason we call them internal Bousfield localizations, as opposed to being enriched Bousfield localizations, which use Hom objects valued in $\mathcal{V}$.

Regarding the Bousfield localization itself, Barwick's construction is an astute one. He does not define a new Bousfield localization. He uses what we refer to as the classical Bousfield localization, that of Hirschhorn in [Hi]. If H is a set of maps we are localizing with respect to, S is a set representing homotopy classes of H, I a generating set of cofibrations with cofibrant domains, he shows his enriched Bousfield localization of a tractable, left proper model category $\mathcal{C}$ is none other than $L_{I \Box S}\mathcal{C}$ in the classical sense, and then uses Theorem 17.4.16 of [Hi] to have his enriched Homs appear. His proof is short, precisely because of his ingenious use of that result, as opposed to ours, which is a tedious reworking of Hirschhorn's work in [Hi] about left Bousfield localization, in the event that Hom sets are replaced by internal Hom objects.

Barwick's work is elegantly generalized in [GR], in the larger setting of combinatorial model categories. Gutierrez and Roitzhem consider a Quillen adjunction of two variables between combinatorial model categories, $\mathcal{C}\times\mathcal{D}\to\mathcal{E}$, and they define the left Bousfield localization $L_S\mathcal{E}$ of $\mathcal{E}$ with respect to a set S of morphisms in $\mathcal{C}$, which they call the S-local model structure on $\mathcal{E}$. In the notations of [B] as used above, if we consider a Quillen adjunction of two variables $\mathcal{C}\otimes\mathcal{V}\to\mathcal{C}$ associated with a $\mathcal{V}$-enrichment, then $L_S\mathcal{C}=L_{I\Box S}\mathcal{C}$, which corresponds to a $\mathcal{V}$-enriched left Bousfield localization of $\mathcal{C}$ with respect to S as defined by Barwick.

Acknowledgments. The author would like to thank the organizers of the "Exchange of Mathematical Ideas - 2018" conference for providing a terrific work environment during which part of this work was done and presented. The author is grateful to J. Gutierrez and D. White for pointing out past work on the present subject.

2 Construction

We fix a universe $\mathbb{U}$. Let $(\mathcal{C},W)$ be a $\mathbb{U}$-small pseudo-model category, $\mathcal{D}_0$ a proper, cellular model category, $\mathcal{D}_0^{\mathcal{C}^{\text{op}}}$ the model category of functors from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}_0$. It is also a proper, cellular model category (Thm 13.1.14 and Prop 12.1.5 of [Hi]). We aim to take left and right Bousfield localizations of $\mathcal{D}_0^{\mathcal{C}^{\text{op}}}$ with respect to certain classes of maps, in a generalized sense. By this we mean we will use internal Hom objects in the definition of homotopy function complexes instead of Hom sets. We will prepare the ground for operating such localizations. This is what "construction" is in reference to. We will first introduce $\Gamma=\{h_y\to h_x \mid x\to y\in W^{\text{op}}\}$, where $h_a=\text{Hom}_{\mathcal{C}}(-,a)$ and W denotes the class of weak equivalences in $\mathcal{C}$. We will first construct $L_\Gamma\mathcal{D}_0^{\mathcal{C}^{\text{op}}}=\mathcal{D}_0^{\mathcal{C}^{\text{op}}\,\wedge}$, the internal left Bousfield localization of $\mathcal{D}_0^{\mathcal{C}^{\text{op}}}$ with respect to Γ. From that point forward, we will introduce a class K_0 of objects in $\mathcal{D}_0^{\mathcal{C}^{\text{op}}\,\wedge}$ and consider the class $\mathcal{K}_0$ of internal K_0-colocal equivalences. We will then construct $R_{\mathcal{K}_0}\mathcal{D}_0^{\mathcal{C}^{\text{op}}\,\wedge}$ the internal right Bousfield localization of $\mathcal{D}_0^{\mathcal{C}^{\text{op}}\,\wedge}$ with respect to $\mathcal{K}_0$.

Many of the results we will discuss in this work are stated and proved in the classical sense in [Hi]. To make comparison with those original statements easier, next to each claim we will put in bracket the original indexation in [Hi] along with “mod”, indicating that we are stating a modified version thereof.

2.1 Preserving equivalences

In a first time, we would like functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}_0$ to map equivalences $u : x \to y$ in $\mathcal{C}^{\mathrm{op}}$ to equivalences $Fx \to Fy$ in $\mathcal{D}_0$. For this purpose we introduce the following class:

$$\Gamma = \{h_y \to h_x \mid x \to y \in W^{\mathrm{op}}\}$$

If we were to use the classical Yoneda lemma, it would suffice to show we have:

$$\mathrm{Hom}(h_x, F) \xrightarrow{\simeq} \mathrm{Hom}(h_y, F)$$

which would make F Γ-local in the terminology of [Hi]. However, F is $\mathcal{D}_0$-valued, not Set-valued, so we use the enriched Yoneda lemma ([K]). In order to do so we start to use internal Homs, hence we would want:

$$\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_x, F) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_y, F)$$

We will actually show:

$$\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h_x}, \hat{F}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h_y}, \hat{F})$$

in the Reedy structure of $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$ ($\tilde{h}$ cofibrant approximation to h, $\hat{F}$ simplicial resolution of F) which will imply F maps equivalences to equivalences.

2.2 Monoidal structure

2.2.1 Definitions

In order to use internal Homs, we first ask that $\mathcal{D}_0$ be a monoidal model category ([Ho]), with internal Hom $\underline{\mathrm{Hom}}_{\mathcal{D}_0}$. For simplicity, every notion of [Hi] using Hom sets that is generalized using internal Homs will be referred to as a generalized or as an internal concept. For instance, we would define:

$$\mathrm{map}(k_0, F) = \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^\wedge}(\tilde{k_0}, \hat{F})$$

as an internal right homotopy function complex (abbr. iRhfC), where $\tilde{k_0}$ is a cofibrant approximation to k_0, and $\hat{F}$ is a simplicial resolution of F.

For later purposes, we make the following definitions:

Definition 2.2.1.1. An internal right homotopy function complex (abbr. iRhfC) is an object of $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$ of the form $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{X}, \hat{Y})$, where $\tilde{X}$ is a cofibrant approximation to X and $\hat{Y}$ is a simplicial resolution of Y.

Definition 2.2.1.2. An internal left homotopy function complex (abbr. iLhfC) is an object of $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$ of the form $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{X}, \hat{Y})$, where $\tilde{X}$ is a cosimplicial resolution of X and $\hat{Y}$ is a fibrant approximation to Y.

Note that $h_x = \mathrm{Hom}_{\mathcal{C}}(-, x)$ is a set-valued functor, not an object of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Hence we ask that $\mathcal{C}$ be a $\mathcal{D}_0$-enriched category([K]). This we can do since $\mathcal{D}_0$ is a monoidal category.

2.2.2 $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is a monoidal model category

We will also need that $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ be a monoidal model category. We define the monoidal structure on $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ point-wise: if $F, G \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, then for any $x \in \mathcal{C}$, $(F \otimes G)(x) = Fx \otimes Gx$. Endow $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ with the injective model structure; equivalences and cofibrations are defined point-wise. This makes the tensor product on $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ a Quillen bifunctor. Indeed, let $\alpha : U \to V$ be a cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $\beta : W \to X$ a cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ as well, we need:

$$\alpha \Box \beta : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$$

to be a cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, trivial if either of α or β is ([Ho]). Since we take the injective model structure on $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, we have to check that pointwise: let $x \in \mathcal{C}^{\mathrm{op}}$. We are looking at:

$$\alpha \Box \beta(x) : (Vx \otimes Wx) \coprod_{Ux \otimes Wx} (Ux \otimes Xx) \to Vx \otimes Xx$$

Now $U \to V$ cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ with the injective model structure means $Ux \to Vx$ cofibration in $\mathcal{D}_0$, and $W \to X$ cofibration means $Wx \to Xx$ cofibration in $\mathcal{D}_0$, and it follows that the above map is a cofibration in $\mathcal{D}_0$ (since it is a monoidal model category), and this for all $x \in \mathcal{C}^{\mathrm{op}}$, hence a cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Since equivalences are defined pointwise, we also have that $\alpha \Box \beta$ is trivial if either of α or β is. The second condition for being a monoidal model category ([Ho]) is that if $Q1 \to 1$ is a cofibrant approximation to the unit 1, then for all X cofibrant, $Q1 \otimes X \to 1 \otimes X$ is a weak equivalence. Here we assume the unit is cofibrant, a condition we will use later. If that is the case, this condition is automatically satisfied. This makes $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ into a monoidal model category. Observe that having $\otimes$ a Quillen bifunctor, if C is cofibrant in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, then $C \otimes - : \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is left Quillen, so preserves cofibrations and trivial cofibrations, a fact that will be very important in the work that follows.

2.2.3 Construction of $\underline{\mathbf{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}$

The internal Hom of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is formally defined as follows: an element $\alpha \in \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F \otimes G, H)$ is defined pointwise: for $x \in \mathcal{C}$, $\alpha(x) : Fx \otimes Gx \to Hx$ in $\mathcal{D}_0$. Now:

$$\mathrm{Hom}_{\mathcal{D}_0}(Fx \otimes Gx, Hx) \cong \mathrm{Hom}_{\mathcal{D}_0}(Fx, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Gx, Hx))$$

so $\alpha(x)$ would correspond to some $\beta(x) \in \mathrm{Hom}_{\mathcal{D}_0}(Fx, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Gx, Hx))$. Letting

$$\underline{\mathrm{Hom}}_{\mathcal{D}_0}(Gx, Hx) = \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(G, H)(x) \tag{2}$$

we have $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(G, H) \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, and with this notation:

$$\mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F \otimes G, H) \cong \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F, \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(G, H))$$

This can be formalized using the language of ends ([McL]). It suffices to write, still using the identification (2):

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F \otimes G, H) &\cong \int_{x \in \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}_0}(Fx \otimes Gx, Hx) \\ &\cong \int_{x \in \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}_0}(Fx, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Gx, Hx)) \\ &= \int_{x \in \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}_0}(Fx, \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(G, H)(x)) \\ &\cong \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F, \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(G, H)) \end{aligned}$$

We can make this more precise. From now on, we will assume that $(\mathcal{D}_0, \otimes)$ is also symmetric. Following [GK], and working in full generality for later purposes, consider $\mathcal{M}$ a closed symmetric monoidal category, which will be $\mathcal{D}_0$ for us. An $\mathcal{M}$-module structure on a category $\mathcal{C}$ is given by an action:

$$\begin{aligned} \otimes : \mathcal{C} \times \mathcal{M} &\to \mathcal{C} \\ (X, M) &\mapsto X \otimes M \end{aligned}$$

This action is closed if we have two functors:

$$\begin{aligned} \mathrm{map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\to \mathcal{M} \\ (X, Y) &\mapsto \mathrm{map}_{\mathcal{C}}(X, Y) \end{aligned}$$

and:

$$\begin{aligned} \exp : \mathcal{M}^{\mathrm{op}} \times \mathcal{C} &\to \mathcal{C} \\ (K, Y) &\mapsto Y^K \end{aligned}$$

in such a manner that we have, for all $K \in \mathcal{M}$, $X, Y \in \mathcal{C}$, the following natural isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes K, Y) \cong \mathrm{Hom}_{\mathcal{M}}(K, \mathrm{map}_{\mathcal{C}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y^K)$$

Now consider the following functor, where I is an indexing set:

$$\begin{aligned} \mathcal{M}^I \times \mathcal{M} &\to \mathcal{M}^I \\ (X, K) &\mapsto X \otimes K \end{aligned}$$

defined by $(X \otimes K)_i = X_i \otimes K$. This defines a $\mathcal{M}$-module structure on $\mathcal{M}^I$. It is closed if we use the following definitions: for $X, Y \in \mathcal{M}^I$, $K \in \mathcal{M}$, define $Y^K = \underline{\mathrm{Hom}}_{\mathcal{M}}(K, Y)$ by $\underline{\mathrm{Hom}}_{\mathcal{M}}(K, Y)_i = \underline{\mathrm{Hom}}_{\mathcal{M}}(K, Y_i)$ and:

$$\mathrm{map}_{\mathcal{M}^I}(X, Y) = \int_i \underline{\mathrm{Hom}}_{\mathcal{M}}(X_i, Y_i)$$

In particular for $\mathcal{M} = \mathcal{D}_0$ and $I = \mathcal{C}^{\mathrm{op}}$, this gives us, for $X, Y \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $K \in \mathcal{D}_0$:

$$\mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(X \otimes K, Y) \cong \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(X, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(K, Y)) \cong \mathrm{Hom}_{\mathcal{D}_0}(K, \mathrm{map}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(X, Y))$$

Coming back to the general case, consider:

$$\begin{aligned} h_i : I &\to \mathcal{M} \\ j &\mapsto h_i(j) = \coprod_{I(i,j)} 1 \end{aligned}$$

where 1 is the unit of $\mathcal{M}$. Define a monoidal structure on $\mathcal{M}^I$ as follows: for $X, Y \in \mathcal{M}^I$, let $(X \otimes Y)_i = X_i \otimes Y_i$, making $(\mathcal{M}^I, \otimes)$ into a symmetric monoidal category, which is furthermore closed, with internal Hom given by:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{M}^I}(X,Y)_i &= \mathrm{map}_{\mathcal{M}^I}(h_i \otimes X, Y) \\ &= \int_{j \in I} \underline{\mathrm{Hom}}_{\mathcal{M}}(h_i(j) \otimes X_j, Y_j) \end{aligned}$$

For $\mathcal{M} = \mathcal{D}_0$ and $I = \mathcal{C}^{\mathrm{op}}$, this gives us the internal Hom in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F,G)(x) &= \mathrm{map}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_x \otimes F, G) \qquad (3) \\ &= \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y) \otimes Fy, Gy) \\ &= \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y), \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Fy, Gy)) \qquad (4) \\ &= \mathrm{map}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_x, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(F-, G-) \end{aligned}$$

We proceed from (4) to get (2) as follows. For any W:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_0}(W, \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F,G)(x)) &= \mathrm{Hom}_{\mathcal{D}_0}(W, \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y), \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Fy, Gy))) \\ &= \int_{y \in \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}_0}(W, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y), \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Fy, Gy))) \\ &= \int \mathrm{Hom}_{\mathcal{D}_0}(W \otimes h_x(y), \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Fy, Gy)) \\ &= \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(W \otimes h_x, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(F-, G-)) \end{aligned}$$

Now we use the fact that if we define the evaluation functor $Ev_i : \mathcal{M}^I \to \mathcal{M}$ by $Ev_i(X) = X_i$ and $F : \mathcal{M} \to \mathcal{M}^I$ by $F_i(M) = h_i \otimes M$, then we have $F_i \dashv Ev_i$, which reads, for $\mathcal{M} = \mathcal{D}_0$ and $I = \mathcal{C}^{\mathrm{op}}$:

$$\mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_x \otimes M, G) \cong \mathrm{Hom}_{\mathcal{D}_0}(M, G(x))$$

using this in the previous computation gives us:

$$\mathrm{Hom}_{\mathcal{D}_0}(W, \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F,G)(x)) = \mathrm{Hom}_{\mathcal{D}_0}(W, \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Fx, Gx))$$

and this being true for all W, we have:

$$\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(F,G)(x) \cong \underline{\mathrm{Hom}}_{\mathcal{D}_0}(Fx, Gx)$$

2.2.4 More on tensor products

As part of the construction, and for later purposes, we use the following result of [GK], that if I is a Reedy category, $\mathcal{M}$ is a cofibrantly generated symmetric monoidal model category with cofibrant unit, then $\mathcal{M}^I$ is a closed symmetric monoidal model category. We use this for $\mathcal{M} = \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge}$, cellular, in particular cofibrantly generated, with a symmetric monoidal structure and a cofibrant unit. It follows that if we take $I = \Delta^{\mathrm{op}}$, then $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge})^{\Delta^{\mathrm{op}}}$ is a symmetric monoidal model category, in particular $\otimes$ is a Quillen bifunctor.

Note that one has to keep track of what tensor products are used, whether they are part of a monoidal structure, or a generalization thereof, for example when we have model structures. For instance:

$$\begin{aligned} \otimes : (\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge})^{\Delta} \times \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge} &\to (\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge})^{\Delta} \\ (\tilde{k_0}, \tilde{D}) &\mapsto \tilde{k_0} \otimes \tilde{D} \end{aligned}$$

defines a $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge}$-module structure on $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge})^{\Delta}$. If $\tilde{D}$ is a cofibrant approximation to D, $\tilde{k_0}$ a cosimplicial resolution of k_0, in writing:

$$\mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge}}(\tilde{D}, \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge}}(\tilde{k_0}, \hat{X})) \cong \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge}}(\tilde{D} \otimes \tilde{k_0}, \hat{X})$$

as provided by Theorem 3.4.2, the tensor product $\tilde{D} \otimes \tilde{k_0}$ makes sense. Here we have used $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge}} = \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}$, for the simple reason that the definition of an internal Hom is peculiar to the monoidal structure, not the model structure, so is preserved after localization. This observation, which we will make more precise, presupposes that $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^{\wedge}$ is a monoidal model category, which we now show.

2.2.5 $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\,\wedge}$ is a monoidal model category

We show $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\,\wedge}$ is a monoidal model category. We use the same tensor product as for $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, and we need that it be a Quillen bifunctor, that is it satisfies the pushout-product axiom. First cofibrations in $L_{\Gamma}\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}} = \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\,\wedge}$ are those of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ as well. Then if in addition either of f or g is a iΓ-local equivalence (Definition 6.2), so is $f\Box g$. Indeed let $f : U \to V$, $g : X \to Y$. Without loss of generality, let's assume f is a trivial cofibration. Then it is a cofibration, and a iΓ-local equivalence by definition, that is $\underline{\mathrm{Hom}}(\tilde{V}, \hat{W}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{U}, \hat{W})$ for W iΓ-local (Definition 6.1). Now iΓ-local equivalences form an ideal class as shown in Section 5, which implies $U \otimes X \to V \otimes X$ is a iΓ-local equivalence as well, hence so is $\tilde{U} \otimes \tilde{X} \to \tilde{V} \otimes \tilde{X}$, and $\otimes$ being a Quillen bifunctor on $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, for $\tilde{X}$ a cofibrant approximation to some object X of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $\tilde{U} \otimes \tilde{X} \to \tilde{V} \otimes \tilde{X}$ is a cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ if we take $\tilde{U} \to \tilde{V}$ a fibrant cofibrant approximation to $U \to V$ that is a cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Hence $\tilde{U} \otimes \tilde{X} \to \tilde{V} \otimes \tilde{X}$ is a cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\,\wedge}$ as well, so it is a iΓ-local equivalence and a cofibration, i.e. a trivial cofibration. Now:

$$
\begin{tikzcd}
\tilde{U} \otimes \tilde{X} \arrow[r, "triv.cof."] \arrow[d] & \tilde{V} \otimes \tilde{X} \arrow[d] \\
\tilde{U} \otimes \tilde{Y} \arrow[r, "triv.cof."'] \arrow[dr, "triv.cof."'] & \tilde{U} \otimes \tilde{Y} \coprod_{\tilde{U}\otimes\tilde{X}} \tilde{V} \otimes \tilde{X} \arrow[d, dotted, "\simeq"] \\
 & \tilde{V} \otimes \tilde{Y}
\end{tikzcd}
$$

where the bottom horizontal map is a trivial cofibration since those are preserved under pushout, $\tilde{U} \otimes \tilde{Y} \to \tilde{V} \otimes \tilde{Y}$ is a trivial cofibration for the same reason that $\tilde{U} \otimes \tilde{X} \to \tilde{V} \otimes \tilde{X}$ is a trivial cofibration, hence the dotted arrow is an equivalence by the 2-3 property.

Finally it suffices to show $\tilde{U}\otimes\tilde{Y} \coprod_{\tilde{U}\otimes\tilde{X}} \tilde{V}\otimes\tilde{X}$ is a cofibrant approximation to $U \otimes Y \coprod_{U\otimes X} V \otimes X$. It suffices to show $\tilde{A} \coprod_{\tilde{B}} \tilde{C} \xrightarrow{\simeq} A \coprod_B C$ in the event that $\tilde{B} \to \tilde{C}$ corresponds to $\tilde{U} \otimes \tilde{X} \to \tilde{V} \otimes \tilde{X}$, trivial cofibration as shown above. Then referring to the commutative diagram below:

$$\begin{tikzcd}
B \arrow[rr] \arrow[dd] & & C \arrow[dd] & \\
& \tilde{B} \arrow[ul, "\simeq"] \arrow[rr] \arrow[dd] & & \tilde{C} \arrow[ul, "\simeq"'] \arrow[dd] \\
A \arrow[rr] & & A \coprod_B C & \\
& \tilde{A} \arrow[ul, "\simeq"] \arrow[rr] & & \tilde{A} \coprod_{\tilde{B}} \tilde{C} \arrow[ul, dotted]
\end{tikzcd}$$

$\tilde{A} \to \tilde{A} \coprod_{\tilde{B}} \tilde{C}$ is a trivial cofibration as pushout so a weak equivalence in particular. In the top square $B \to C$ is an equivalence by the 2-3 property, $C \to A \coprod_B C$ cofibration, $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ left proper, so $A \to A \coprod_B C$ equivalence as pushout of an equivalence along a cofibration. In the bottom square it follows using the 2-3 property that $\tilde{A} \coprod_{\tilde{B}} \tilde{C} \to A \coprod_B C$ is an equivalence.

Then we also need that for any X cofibrant in $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $Q1 \otimes X \to 1 \otimes X$ is a weak equivalence in $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, i.e. a i$\Gamma$-local equivalence. X cofibrant in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}$ implies that it is cofibrant in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ as well. Then since $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is a monoidal model category, $Q1 \otimes X \xrightarrow{\simeq} 1 \otimes X$, but equivalences are also iΓ-local equivalences by Proposition 4.3.1, hence it is a weak equivalence in $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ as well. We conclude $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is a monoidal model category, with the same tensor product $\otimes$ as $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, and $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}} = \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}$.

2.3 Covers

We regard $\mathcal{D}_0$ as the first of a chain of proper cellular, symmetric monoidal model categories, the idea being that we are interested in maps between localizations of $\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}}$. Denote by $h_i : \mathcal{D}_i \to \mathcal{D}_{i+1}$ a map of model categories, that is, a left Quillen functor. It induces:

$$h_i^{\mathcal{C}^{\mathrm{op}}} : \mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}}$$

pointwise. It follows from Proposition 11.6.5 of [Hi] that since our model categories are cellular, in particular cofibrantly generated, this map is also a left Quillen functor. We refer the reader to Definition 8.5.11 of [Hi] for defining the left derived functor $\mathbb{L}F$ of a left adjoint F, part of a Quillen adjunction. Recall that $\mathcal{C}$ is $\mathcal{D}_0$-enriched, which means it is also $\mathcal{D}_i$-enriched by composition with the h_i's. Define:

$$h^{(i)} = h_{i-1} \circ \cdots \circ h_0$$

and

$$h_x^{(i)} = h^{(i-1)} \circ h_x$$

Let Γ_i be the following class of maps in $\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}}$:

$$\Gamma_i = \{h_y^{(i)} \to h_x^{(i)} \mid x \to y \in W(\mathcal{C}^{\mathrm{op}})\}$$

Define:

$$\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i = \{\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}(g) \mid g \in \Gamma_i\}$$

then by Theorem 3.3.20 of [Hi], we have a left Quillen functor:

$$h_i^{\mathcal{C}^{\mathrm{op}}} : L_{\Gamma_i}\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \to L_{\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i}\mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}}$$

Observe that we have $\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i \subset \Gamma_{i+1}$. Indeed, for $g \in \Gamma_i$:

$$\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}(g) = h_i^{\mathcal{C}^{\mathrm{op}}}(\tilde{C}_i(g)) = h_i \circ \tilde{C}_i(g)$$

$\tilde{C}_i$ cofibrant approximation functor. But we may as well enlarge Γ_i to also contain $\tilde{C}_i(g)$'s since we use internal homotopy function complexes, and its image by h_i is therefore in Γ_{i+1} by definition. It follows i$\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i$-local objects contain i$\Gamma_{i+1}$-local objects, and consequently i$\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i$-local equivalences are contained in iΓ_{i+1}-local equivalences. We are looking at the following picture:

$$\begin{tikzcd}
\mathcal{D}_i \arrow[r, "h_i"] \arrow[d, rightsquigarrow] & \mathcal{D}_{i+1} \arrow[d, rightsquigarrow] \\
\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \arrow[r, "h_i^{\mathcal{C}^{\mathrm{op}}}"] \arrow[d, "L_{\Gamma_i}"'] & \mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}} \arrow[d, "L_{\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i}"] \\
L_{\Gamma_i}\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \arrow[r, "h_i^{\mathcal{C}^{\mathrm{op}}}"] & L_{\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i}\mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}}
\end{tikzcd}$$

If we introduce the right Quillen adjoint $k_{i+1}^{\mathcal{C}^{\mathrm{op}}}$ of $h_i^{\mathcal{C}^{\mathrm{op}}}$, then one can define its right derived form following Definition 8.5.11 of [Hi], and Theorem 3.3.20

again gives us that we have a Quillen adjunction (here we assume $Ł_{\Gamma_i}\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}}$ is right proper):

$$R_{\mathbb{R}k_{i+1}^{\mathcal{C}^{\mathrm{op}}}\mathcal{K}_{i+1}} L_{\Gamma_i}\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \underset{k_{i+1}^{\mathcal{C}^{\mathrm{op}}}}{\overset{h_i^{\mathcal{C}^{\mathrm{op}}}}{\rightleftarrows}} R_{\mathcal{K}_{i+1}} L_{\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i}\mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}}$$

induced from:

$$\begin{tikzcd}
L_{\Gamma_i}\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \arrow[r, "h_i^{\mathcal{C}^{\mathrm{op}}}"] \arrow[d, "R_{\mathbb{R}k_{i+1}^{\mathcal{C}^{\mathrm{op}}}\mathcal{K}_{i+1}}"'] & L_{\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i}\mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}} \arrow[d, "R_{\mathcal{K}_{i+1}}"] \\
R_{\mathbb{R}k_{i+1}^{\mathcal{C}^{\mathrm{op}}}\mathcal{K}_{i+1}} L_{\Gamma_i}\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}} \arrow[r, "h_i^{\mathcal{C}^{\mathrm{op}}}"'] & R_{\mathcal{K}_{i+1}} L_{\mathbb{L}h_i^{\mathcal{C}^{\mathrm{op}}}\Gamma_i}\mathcal{D}_{i+1}^{\mathcal{C}^{\mathrm{op}}}
\end{tikzcd}$$

where K_i is a class of objects of $\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}\wedge}$ and $\mathcal{K}_i = \{K_i-\text{colocal equivalences }\}$. This presupposes that the left localization of $\mathcal{D}_i^{\mathcal{C}^{\mathrm{op}}}$ is right proper, cellular. That it is cellular is shown when we prove that we have an internal left Bousfield localization, but whether it is right proper is not automatic. This is something we have to assume. In other terms this construction holds for symmetric monoidal model categories $\mathcal{D}_i$ whose internal left Bousfield localizations are right proper.

3 Foundational results

3.1 Elementary results

We list a few results that will be useful in the sequel. First we have:

$$\underline{\mathrm{Hom}}(1, A) \cong A$$

since for any W, $\mathrm{Hom}(W, \underline{\mathrm{Hom}}(1, A)) \cong \mathrm{Hom}(W \otimes 1, A) \cong \mathrm{Hom}(W, A)$. We also have:

$$\mathrm{Hom}(1, \underline{\mathrm{Hom}}(A, B)) \cong \mathrm{Hom}(A, B)$$

as it directly follows from the adjunction isomorphism $\mathrm{Hom}(1, \underline{\mathrm{Hom}}(A, B)) \cong \mathrm{Hom}(1 \otimes A, B) \cong \mathrm{Hom}(A, B)$. From this we have the very useful:

Proposition 3.1.1. Internal equivalences $\underline{\mathrm{Hom}}(\tilde{A}, \hat{B}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{A}, \hat{C})$ imply classical equivalences $\mathrm{Hom}(\tilde{A}, \hat{B}) \xrightarrow{\simeq} \mathrm{Hom}(\tilde{A}, \hat{C})$.

Proof. It suffices to write:

$$\begin{tikzcd}
\mathrm{Hom}(1, \underline{\mathrm{Hom}}(\tilde{A}, \hat{B})) \arrow[r, "\simeq"] \arrow[d, equal] & \mathrm{Hom}(1, \underline{\mathrm{Hom}}(\tilde{A}, \hat{C})) \arrow[d, equal] \\
\mathrm{Hom}(\tilde{A}, \hat{B} \arrow[r, dotted, "\simeq"'] & \mathrm{Hom}(\tilde{A}, \hat{C})
\end{tikzcd}$$

□

A very important result is the following, a modified version of Theorem 17.7.7 of [Hi]

Theorem[17.7.7 mod] 3.1.2. Let $g : X \to Y$ a map in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Then g is a weak equivalence if and only if for all W $g_* : \underline{\mathrm{Hom}}(\tilde{W}, \hat{X}) \to \underline{\mathrm{Hom}}(\tilde{W}, \hat{Y})$ is a weak equivalence in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$, if and only if for all Z in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $g^* : \underline{\mathrm{Hom}}(\tilde{Y}, \hat{Z}) \to \underline{\mathrm{Hom}}(\tilde{X}, \hat{Z})$ is a weak equivalences in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$.

Proof. Suppose $g : X \to Y$ is an equivalence, then by the original Theorem 17.7.7 of [Hi], we have the following top equivalence for any object W:

$$\begin{tikzcd}
\mathrm{Hom}(\widetilde{W \otimes A}, \hat{X}) \arrow[r, "\simeq"] \arrow[d, equal] & \mathrm{Hom}(\widetilde{W \otimes A}, \hat{Y}) \arrow[d, equal] \\
\mathrm{Hom}(\tilde{W} \otimes \tilde{A}, \hat{X}) \arrow[r, "\simeq"] \arrow[d, "\cong"'] & \mathrm{Hom}(\tilde{W} \otimes \tilde{A}, \hat{Y}) \arrow[d, "\cong"] \\
\mathrm{Hom}(\tilde{W}, \underline{\mathrm{Hom}}(\tilde{A}, \hat{X})) \arrow[r, dotted, "\simeq"'] & \mathrm{Hom}(\tilde{W}, \underline{\mathrm{Hom}}(\tilde{A}, \hat{Y}))
\end{tikzcd} \tag{5}$$

and this being true for any W it follows $\underline{\mathrm{Hom}}(\tilde{A}, \hat{X}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{A}, \hat{Y})$. Conversely, if that is true, that implies by Proposition 3.1.1 $\mathrm{Hom}(\tilde{A}, \hat{X}) \xrightarrow{\simeq} \mathrm{Hom}(\tilde{A}, \hat{Y})$, hence an equivalence $g : X \xrightarrow{\simeq} Y$ by the original Theorem 17.7.7. For the second part of the Theorem we proceed in like manner. For any W we have:

$$\begin{tikzcd}
\mathrm{Hom}(\widetilde{W \otimes Y}, \hat{Z}) \arrow[r, "\simeq"] \arrow[d, equal] & \mathrm{Hom}(\widetilde{W \otimes X}, \hat{Z}) \arrow[d, equal] \\
\mathrm{Hom}(\tilde{W} \otimes \tilde{Y}, \hat{Z}) \arrow[d, "\cong"] & \mathrm{Hom}(\tilde{W} \otimes \tilde{X}, \hat{Z}) \arrow[d, "\cong"] \\
\mathrm{Hom}(\tilde{W}, \underline{\mathrm{Hom}}(\tilde{Y}, \hat{Z})) \arrow[r, dotted, "\simeq"] & \mathrm{Hom}(\tilde{W}, \underline{\mathrm{Hom}}(\tilde{X}, \hat{Z}))
\end{tikzcd}$$

The reasoning for showing that g is an equivalence if and only if if g^* is an equivalence is identical to the one above for showing that g is an equivalence if and only if g_* is one as well. $\square$

3.2 Yoneda

We take the left Bousfield localization of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ with respect to $\Gamma = \{h_y \to h_x \mid x \to y \in W(\mathcal{C}^{\mathrm{op}})\}$. $F \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is iΓ-local (see Definition 6.1) if it satisfies:

$$\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h_x}, \hat{F}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h_y}, \hat{F})$$

At the same time, recall that we initially wanted to have functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}_0$ to map equivalences to equivalences. Thus we need a Yoneda lemma for enriched functors. We use [K] and [McL]. Consider a $\mathcal{D}_0$-functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}_0$. Let $x \in \mathcal{C}$. Then the morphism:

$$\phi_y : Fx \to \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y), Fy)$$

expresses Fx as the end $\int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y), Fy)$. Observe that we could have obtained this result using the $\mathcal{D}_0$-module structure on $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$: We have:

$$\mathrm{map}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_x, F) = \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y), Fy)$$

but we also have, with (3):

$$\begin{aligned}
\mathrm{map}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_x, F) &= \mathrm{map}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(h_x \otimes 1, F) \\
&= \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(1, F)(x) \\
&= F(x)
\end{aligned}$$

hence:

$$F(x) = \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(h_x(y), Fy)$$

We improve on this result since we still have to connect this to a iRhfC.

Theorem 3.2.1.

$$Fx \cong \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(\tilde{h}_x(y), Fy) \tag{6}$$

Proof. This is an adaptation of the proof found in [K]. Consider any $\mathcal{D}_0$-natural map $\alpha_y^c : X \to \underline{\mathrm{Hom}}(\tilde{h}_x(y), Fy)$ whose adjoint we will represent by the same letter: $\alpha_y^c : \tilde{h}_x(y) \to \underline{\mathrm{Hom}}(X, Fy)$. We have $\phi_y^c : Fx \to \underline{\mathrm{Hom}}(\tilde{h}_x(y), Fy)$, with adjoint again represented by the same letter, $\phi_y^c : \tilde{h}_x(y) \to \underline{\mathrm{Hom}}(Fx, Fy)$. We want to show there is a unique $\eta : X \to Fx$ such that $\alpha_y^c = \phi_y^c \eta$, and that will prove our result. For the adjoint, that amounts to showing there exists a unique η such that the triangular diagram below commutes:

$$\begin{tikzcd}
\alpha_y : h_x(y) \arrow[r] & \underline{\mathrm{Hom}}(Fx, Fy) \arrow[r, "\mathrm{Hom}(\eta, id)"] & \mathrm{Hom}(X, Fy) \\
\tilde{h}_x(y) \arrow[u, "\simeq"'] \arrow[ur, dotted, "\phi_y^c"] \arrow[urr, dotted, "\alpha_y^c"'] & &
\end{tikzcd}$$

but from the classical proof we know there is such a η, and it is unique, hence by composition we have our result.

$\square$

3.3 Localization with respect to Γ

Recall that we aim to have functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}_0$ that map equivalences to equivalences. This is implemented using a Bousfield localization with respect to $\Gamma = \{h_y \to h_x \mid x \to y \in W(\mathcal{C}^{\mathrm{op}})\}$. Recall that $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}_0$ is iΓ-local if for any element of Γ, we have:

$$\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_x, \hat{F}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_y, \hat{F}) \tag{7}$$

where $\tilde{h}$ is a cofibrant approximation of h, and $\hat{F}$ is a simplicial resolution of F. Recall that for simplicial objects $\hat{X}$, we define the simplicial set $\underline{\mathrm{Hom}}(A, \hat{X})$, as in [Hi]:

$$\underline{\mathrm{Hom}}(A, \hat{X})_n = \underline{\mathrm{Hom}}(A, \hat{X}_n)$$

We show (7) implies $Fx \xrightarrow{\simeq} Fy$ if $x \to y$ is in W if F is iΓ-local. We use the enriched Yoneda lemma (6):

$$Fx \cong \int_{y \in \mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(\tilde{h}_x(y), Fy)$$

Since $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_x, \hat{F}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_y, \hat{F})$ is an equivalence in the Reedy model structure of $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$, this means for all $[n] \in \Delta$, we have:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_x, \hat{F})_n \arrow[r, "\simeq"] \arrow[d, equal] & \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_y, \hat{F})_n \arrow[d, equal] \\
\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_x, \hat{F}_n) \arrow[r, dotted, "\simeq"] & \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{h}_y, \hat{F}_n)
\end{tikzcd} \tag{8}$$

$\hat{F}$ being a simplicial resolution of F means $cs_*(F) \xrightarrow{\simeq} \hat{F}$, where cs_* is the constant simplicial functor. This being a Reedy weak equivalence in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$, it follows that for all $[n] \in \Delta$, we have equivalences in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$: $F = (cs_*(F))_n \xrightarrow{\simeq} \hat{F}_n$. It follows from (8):

$$\begin{tikzcd}
\hat{F}_n(x) \cong \int_{t\in\mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(\tilde{h}_x(t), \hat{F}_n(t)) \arrow[r, "\simeq"] & \hat{F}_n(y) \cong \int_{t\in\mathcal{C}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{D}_0}(\tilde{h}_y(t), \hat{F}_n(t)) \\
F(x) \arrow[u, "\simeq", "\mathcal{D}_0"'] \arrow[r, dotted, "\simeq"'] & F(y) \arrow[u, "\simeq", "\mathcal{D}_0"']
\end{tikzcd}$$

where the bottom equivalence follows from the 2-3 property. This shows iΓ-local objects map equivalences to equivalences.

3.4 $\underline{\mathbf{Hom}}(\tilde{X}, \hat{Y})$ is fibrant

Crucial in all our work is the fact that internal homotopy function complexes are fibrant objects. This is needed to use Theorem 17.7.7 to prove its modified version, Theorem 3.1.2. We state this as a result:

Proposition 3.4.1. Let C be cofibrant in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $\hat{X}$ a simplicial resolution of $X \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Then $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(C, \hat{X})$ is fibrant in the Reedy model structure of $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$.

Proof. $\underline{\mathrm{Hom}}(C, \hat{X})$ fibrant means $\underline{\mathrm{Hom}}(C, \hat{X}) \to *$ is a fibration in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$, where $*$ denotes the terminal object of $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$. This is true if for all $[n] \in \Delta$, we have:

$$\underline{\mathrm{Hom}}(C, \hat{X})_n = \underline{\mathrm{Hom}}(C, \hat{X}_n) \to * \times_{M_n *} M_n \underline{\mathrm{Hom}}(C, \hat{X})$$

is a fibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, where M_nA is the n-th matching object of A (see [Hi]). Given the definition of the matching object, the object on the right of this map simplifies as $M_n\underline{\mathrm{Hom}}(C,\hat{X})$, wich is equal to $\underline{\mathrm{Hom}}(C,M_n\hat{X})$, since the matching object is a limit, and the internal Hom commutes with limits. Thus we seek to show that:

$$\underline{\mathrm{Hom}}(C,\hat{X}_n) \to \underline{\mathrm{Hom}}(C,M_n\hat{X})$$

is a fibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Let $D \to E$ be a trivial cofibration in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. We need a lift α in the commutative diagram below:

$$\begin{tikzcd} D \arrow[r] \arrow[d, "triv\ cof"'] & \underline{\mathrm{Hom}}(C,\hat{X}_n) \arrow[d] \\ E \arrow[r] \arrow[ur, dotted, "\alpha"] & \underline{\mathrm{Hom}}(C,M_n\hat{X}) \end{tikzcd}$$

by adjunction this is equivalent to having a lift in the diagram below:

$$\begin{tikzcd} D\otimes C \arrow[r] \arrow[d] & X_n \arrow[d] \\ E\otimes C \arrow[r] \arrow[ur, dotted] & M_n\hat{X} \end{tikzcd}$$

but C is cofibrant, $\otimes$ is a left Quillen functor, so $D\otimes C \to E\otimes C$ is a trivial cofibration, and $\hat{X}$ being a simplicial resolution, it is fibrant in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$, so we have such a lift, which completes the proof. $\square$

The other internal homotopy function complex we work with is the one we consider after having taken a left Bousfield localization with respect to Γ, and we find ourselves in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge} = L_\Gamma\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. The iLhfC of interest is now $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}}(\tilde{k_0},\hat{F})$, where $k_0 \in K_0$, $\tilde{k_0}$ is a cosimplicial resolution of k_0, and $\hat{F}$ is a fibrant approximation to F. As a preliminary result, we prove:

Theorem 3.4.2.

$$\mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}}(\tilde{D},\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}}(\tilde{k_0},\hat{X})) \cong \mathrm{Hom}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}}(\tilde{D}\otimes\tilde{k_0},\hat{X})$$

Proof. Since for $\tilde{k_0}$ a cosimplicial resolution, we have $\underline{\mathrm{Hom}}(\tilde{k_0},A)_n = \underline{\mathrm{Hom}}(\tilde{k_0}_n,A)$, and for $\hat{W}$ a simplicial resolution we have $\underline{\mathrm{Hom}}(Y,\hat{W})_n = \underline{\mathrm{Hom}}(Y,\hat{W}_n)$, it suffices to show this isomorphism of simplicial sets on components:

$$\begin{aligned}\mathrm{Hom}(\tilde{D}, \underline{\mathrm{Hom}}(\tilde{k_0}, \hat{X}))_n &= \mathrm{Hom}(\tilde{D}, \underline{\mathrm{Hom}}(\tilde{k_0}, \hat{X})_n) \\ &= \mathrm{Hom}(\tilde{D}, \underline{\mathrm{Hom}}(\tilde{k_0}_n, \hat{X})) \\ &\cong \mathrm{Hom}(\tilde{D} \otimes \tilde{k_0}_n, \hat{X}) \\ &= \mathrm{Hom}((\tilde{D} \otimes \tilde{k_0})_n, \hat{X}) \\ &= \mathrm{Hom}(\tilde{D} \otimes \tilde{k_0}, \hat{X})_n\end{aligned}$$

$\square$

Proposition 3.4.3. For $\tilde{k_0}$ a cofibrant approximation to k_0, $\hat{F}$ a simplicial resolution of F, objects of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^\wedge$, $\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^\wedge}(\tilde{k}_0, \hat{F})$ is fibrant in the Reedy model structure of $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}{}^\wedge)^{\Delta^{\mathrm{op}}}$.

Proof. It follows from the adjunction isomorphism, that the adjoint to:

$$\begin{tikzcd} h_0 \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\tilde{k_0}, \hat{X}) \arrow[d] \\ l_0 \arrow[r] & \underline{\mathrm{Hom}}(\tilde{t_0}, \hat{X}) \times_{\underline{\mathrm{Hom}}(\tilde{t_0}, \hat{Y})} \underline{\mathrm{Hom}}(\tilde{k_0}, \hat{Y}) \end{tikzcd}$$

is the following diagram:

$$\begin{tikzcd} \tilde{t_0} \otimes l_0 \coprod_{\tilde{t_0} \otimes h_0} \tilde{k_0} \otimes h_0 \arrow[r] \arrow[d] & \hat{X} \arrow[d] \\ \tilde{k_0} \otimes l_0 \arrow[r] & \hat{Y} \end{tikzcd}$$

In particular if L_n denotes the nth-latching object functor,

$$\begin{tikzcd} \tilde{C} \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\tilde{k_{0,n}}, \hat{X}) \arrow[d] \\ \tilde{D} \arrow[r] & \underline{\mathrm{Hom}}(L_n\tilde{k_0}, \hat{X}) \times_{\underline{\mathrm{Hom}}(L_n\tilde{k_0}, \hat{Y})} \underline{\mathrm{Hom}}(\tilde{k_{0,n}}, \hat{Y}) \end{tikzcd}$$

which simplifies to:

$$\begin{tikzcd} \tilde{C} \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\tilde{k_0}, \hat{X}) \arrow[d] \\ \tilde{D} \arrow[r] & M_n\underline{\mathrm{Hom}}(\tilde{k_0}, \hat{X}) \end{tikzcd}$$

if $\hat{Y} = *$, has:

$$\begin{array}{ccc} L_n\tilde{k_0} \otimes \tilde{D} \coprod_{L_n\tilde{k_0}\otimes\tilde{C}} \tilde{k_{0,n}} \otimes \tilde{C} & \longrightarrow & \hat{X} \\ \downarrow & & \downarrow \\ \tilde{k_{0,n}} \otimes \tilde{D} & \longrightarrow & \star \end{array}$$

for adjoint. Now the map on the right is a fibration. If the one on the left is a trivial cofibration, we have a lift, and that would prove our claim. But $\tilde{C} \to \tilde{D}$ is a cofibration, $\otimes$ is a Quillen bifunctor since $(\mathcal{D}_0^{\mathcal{C}^{op}} \wedge)^{\Delta^{op}}$ is a symmetric monoidal model category as argued in Section 2.2.4, hence $\tilde{k_0}$ being cofibrant in $(\mathcal{D}_0^{\mathcal{C}^{op}} \wedge)^{\Delta^{op}}$ as a cosimplicial resolution, the functor $\tilde{k_0} \otimes - : \mathcal{D}_0^{\mathcal{C}^{op}} \wedge \to (\mathcal{D}_0^{\mathcal{C}^{op}} \wedge)^{\Delta^{op}}$ is a left Quillen functor, so preserves trivial cofibrations, hence $\tilde{k_0} \otimes \tilde{C} \to \tilde{k_0} \otimes \tilde{D}$ is a trivial cofibration in $(\mathcal{D}_0^{\mathcal{C}^{op}} \wedge)^{\Delta^{op}}$, which means exactly that the left vertical map in the above commutative diagram is a trivial cofibration in $\mathcal{D}_0^{\mathcal{C}^{op}} \wedge$ by definition of cofibrations in Reedy model categories. This completes our proof. $\square$

3.5 Simplicial resolution

In this subsection, we show that $\underline{\mathrm{Hom}}(A, \hat{X})$ is a simplicial resolution of $\underline{\mathrm{Hom}}(A, X)$ if $\hat{X}$ is a simplicial resolution of X. This fact is implied in the proof of Proposition 3.1.2. This follows from Proposition 17.4.16 of [Hi], which states that if $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen adjunction, X is cofibrant in $\mathcal{M}$, Y is fibrant in $\mathcal{N}$, $\hat{Y}$ is a simplicial resolution of Y, then $G(\hat{Y})$ is a simplicial resolution of $G(Y)$. We apply this to the case:

$$- \otimes A : \mathcal{D}_0^{\mathcal{C}^{op}} \rightleftarrows \mathcal{D}_0^{\mathcal{C}^{op}} : \underline{\mathrm{Hom}}(A, -) = G$$

Let $\hat{Y}$ be a simplicial resolution of Y fibrant. Then $G(\hat{Y}) = \underline{\mathrm{Hom}}(A, \hat{Y})$ is a simplicial resolution of $G(Y) = \underline{\mathrm{Hom}}(A, Y)$. For any Y, take a fibrant replacement of $\underline{\mathrm{Hom}}(A, Y)$:

$$\begin{aligned} \underline{\mathrm{Hom}}(A, Y) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}(A, Y) &= RG(Y) \\ &= G(RY) = \underline{\mathrm{Hom}}(A, RY) \end{aligned}$$

so that we have $\underline{\mathrm{Hom}}(A, Y) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(A, RY)$, which implies that $cs_*\underline{\mathrm{Hom}}(A, Y) \xrightarrow{\simeq} cs_*\underline{\mathrm{Hom}}(A, RY) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(A, \hat{Y})$. Here $\hat{Y}$ is a simplicial resolution of RY. But $Y \xrightarrow{\simeq} RY$ implies $cs_*Y \xrightarrow{\simeq} cs_*RY \xrightarrow{\simeq} \hat{Y}$, so $\hat{Y}$ is also a simplicial resolution of Y, and we have that $\underline{\mathrm{Hom}}(A, \hat{Y})$ is a simplicial resolution of $\underline{\mathrm{Hom}}(A, Y)$ as claimed. In the same manner we would show that if $\tilde{Y}$ is a cosimplicial resolution of Y, then $\underline{\mathrm{Hom}}(\tilde{Y}, B)$ is a simplicial resolution of $\underline{\mathrm{Hom}}(Y, B)$.

3.6 Equivalence of internal homotopy function complexes

Our internal homotopy function complexes are defined as the homotopy function complexes of [Hi], save that instead of using Hom sets, we use internal Homs. We want those internal homotopy function complexes to be independent of the choice of resolutions and approximations used in defining them. We first need a couple of definitions, variants of those found in [Hi]:

Definition 3.6.1. A change of iRhfC map:

$$(\tilde{X}, \hat{Y}, \underline{\mathrm{Hom}}(\tilde{X}, \hat{Y})) \to (\tilde{X}', \hat{Y}', \underline{\mathrm{Hom}}(\tilde{X}', \hat{Y}'))$$

is a triple (f, g, h) formed of a map of cofibrant approximations $f : \tilde{X} \to \tilde{X}'$, a map of simplicial resolutions $g : \hat{Y} \to \hat{Y}'$ and the map of simplicial objects $h : \underline{\mathrm{Hom}}(\tilde{X}, \hat{Y}) \to \underline{\mathrm{Hom}}(\tilde{X}', \hat{Y}')$ induced by f and g.

Definition 3.6.2. For $X, Y \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, we define the category $iRhfC(X, Y)$ to be the category of iRhfC's from X to Y and with changes of iRhfC maps as morphisms.

Theorem 3.6.3. Let $X, Y \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Then any two iRhfC's from X to Y are connected by an essentially unique zig-zag of changes of iRhfC maps.

Proof. This follows from Thm 14.4.5 of [Hi], which states that if $\mathcal{C}$ is a category, $X, Y \in \mathcal{C}$, $B\mathcal{C}$ is contractible, then there exists an essentially unique zig-zag from X to Y in $\mathcal{C}$, and the proposition that follows, a modified version of Proposition 17.2.10 of [Hi] to the internal setting. □

Proposition 3.6.4. Let $X, Y \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Then $BiRhfC(X, Y)$ is contractible.

Proof. We have $BiRhfC(X, Y) = BsRes(\underline{\mathrm{Hom}}(\tilde{X}, Y))$ by Section 3.5, where $sRes$ stands for simplicial resolution, and this latter category is contractible by Proposition 16.1.5 of [Hi] □

4 The cardinal γ in the proof of Proposition 4.5.1

To prove that we have an internal left Bousfield localization, we use Theorem 11.3.1 of [Hi], which itself needs Proposition 4.5.1 that we generalize to our setting. The proof of the latter proposition in the classical case uses a cardinal γ. Following Definition 4.5.3 of [Hi], $\gamma = \xi^{\xi}$, where ξ is the smallest cardinal that is at least as large as any of the cardinals that are the subject of the following five sections. We use the same definitions in the generalized case.

4.1 Size of the cells of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$

By definition, the size of the cells of a cellular model category $\mathcal{M}$ is the smallest cardinal for which Theorem 12.3.1 of [Hi] holds. This theorem makes no use of a notion of equivalence, and can be used as is, hence holds also in the internal setting. Hence we can define the size of the cells of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ following that result.

4.2 Compactness of the domains of I

$\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is a cellular model category, in particular cofibrantly generated, and let I denotes its set of generating cofibrations. Let η, as in [Hi], be a cardinal such that the domains of elements of I are η-compact.

4.3 Cardinal λ in the proof of Theorem 4.3.1

As in [Hi], let λ be the cardinal used in the proof of Theorem 4.3.1. This result invokes a set $\widetilde{\Lambda\Gamma}$, originally introduced in Proposition 4.2.5 of [Hi], which uses equivalences, and therefore needs to be stated and proved in the internal setting. To prove it, we invoke the equivalence between Γ-local equivalences and iΓ-local equivalences (Lemma 5.11). In the proof of the original Proposition, one result is interesting in its own right, and we prove it in the internal case, it is Proposition 4.3.1 below. We need it in Section 2.2.5.

Proposition[3.1.5 mod] 4.3.1. Let Γ be a class of maps in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. Then any weak equivalence in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is also a iΓ-local equivalence.

Proof. Let $A \to B$ be a weak equivalence in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. If X is a iΓ-local object, we want $\underline{\mathrm{Hom}}(\tilde{B}, \hat{X}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{A}, \hat{X})$, $\tilde{A}$ a cofibrant approximation to A, and $\hat{X}$ a simplicial resolution to X. For C cofibrant in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, we have:

$$\begin{tikzcd}
\mathrm{Hom}(C, \underline{\mathrm{Hom}}(\tilde{B}, \hat{X})) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(C, \underline{\mathrm{Hom}}(\tilde{A}, \hat{X})) \arrow[d, "\cong"] \\
\mathrm{Hom}(C \otimes \tilde{B}, \hat{X}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(C \otimes \tilde{A}, \hat{X}) \arrow[d, equal] \\
\mathrm{Hom}(\widetilde{C \otimes B}, \hat{X}) \arrow[r, dotted, "\simeq"'] & \mathrm{Hom}(\widetilde{C \otimes A}, \hat{X})
\end{tikzcd}$$

In the last line, we have used the fact that $\widetilde{C \otimes A} = \tilde{C} \otimes \tilde{A}$ since:

$$\begin{tikzcd}
C \otimes \tilde{A} \arrow[r, "\simeq"] & C \otimes A \\
\tilde{C} \otimes \tilde{A} \arrow[u, "\simeq"] \arrow[ur, dotted, "\simeq"'] &
\end{tikzcd}$$

and we have used the fact that $\otimes$ being a left Quillen functor, C being cofibrant, if $\tilde{A} \to \tilde{B}$ is a fibrant cofibrant approximation to $A \to B$ that is a cofibration, then $C \otimes \tilde{A} \to C \otimes \tilde{B}$ is a trivial cofibration as well, in particular it is a weak equivalence, so by the original result of [Hi], it is a Γ-local equivalence, hence $\widetilde{C \otimes A} \to \widetilde{C \otimes B}$ is a Γ-local equivalence so that the bottom horizontal map above is an equivalence, since X iΓ-local is also Γ-local by Lemma 5.10. It follows from the above commutative diagram and Theorem 17.7.7 of [Hi] that we have an equivalence $\underline{\mathrm{Hom}}(\tilde{B}, \hat{X}) \to \underline{\mathrm{Hom}}(\tilde{A}, \hat{X})$ in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$, that is $A \to B$ is a iΓ-local equivalence. $\square$

Proposition[4.2.5 mod] 4.3.2. If I denotes the set of generating cofibrations of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, Γ is a class of maps in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, then there exists a set $\widetilde{\Lambda\Gamma}$ of relative I-cell complexes whose domains are cofibrant, such that every element of $\widetilde{\Lambda\Gamma}$ is a iΓ-local equivalence, and an object W is iΓ-local if and only if $W \to *$ is a $\widetilde{\Lambda\Gamma}$-injective.

Proof. By Lemmas 5.10 and 5.11, this is equivalent to the original Proposition. $\square$

The cardinal in Theorem 4.3.1 of [Hi] is $\lambda = succ(\kappa)$, κ a cardinal which according to Lemma 10.4.6 of [Hi] is such that the domain of every element of $\widetilde{\Lambda\Gamma}$ is κ-small relative to the subcategory of relative $\widetilde{\Lambda\Gamma}$-cell complexes. This is a classical notion and needs not be generalized.

4.4 Cardinal κ in Proposition 12.5.3 of [Hi]

This proposition is applied to the set $\widetilde{\Lambda\Gamma}$. It mentions a cardinal κ at least as large as four kinds of cardinals, two of which are cardinals given by Proposition 12.5.2 of [Hi], which makes use of a Hom set. We generalize this Proposition presently:

Proposition[12.5.2 mod] 4.4.1. If X is a cofibrant object of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, then there is a cardinal η such that for $\nu \geq 2$ a cardinal, Y a cell complex of size ν, $\underline{\mathrm{Hom}}(X, Y)$ has cardinality at most ν^η.

Proof. Let C be cofibrant in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. We have $\mathrm{Hom}(C, \underline{\mathrm{Hom}}(X, Y)) \cong \mathrm{Hom}(C \otimes X, Y)$. Since we are in a monoidal model category, $C \otimes X$ is again cofibrant. We apply the original Proposition of [Hi] to $C \otimes X$, cofibrant, and Y, which gives $size(\mathrm{Hom}(C \otimes X, Y)) \leq \nu^\eta$. Finally, $size(\underline{\mathrm{Hom}}(X, Y)) < size(\mathrm{Hom}(C, \underline{\mathrm{Hom}}(X, Y)) = size(\mathrm{Hom}(C \otimes X, Y))$, which completes the proof. $\square$

4.5 Cardinal κ in Proposition 12.5.7 of [Hi]

κ is an infinite cardinal at least as large as four types of cardinals, two of which are given by Proposition 4.4.1 above, and one of which is given by Definition 12.5.5 of [Hi], which invokes a smallness argument, hence does not need to be modified.

5 Results needed for an internal left Bousfield Localization

The following result is the first one that is needed to prove that we do have an internal left Bousfield localization:

Proposition[3.2.3 mod] 5.1. For Γ a class of maps in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, the class of i$\Gamma$-local equivalences satisfies the 2-3 property.

Proof. This is just a generalization of Hirschhorn's proof in [Hi]: let $g : X \to Y$, $h : Y \to Z$ be maps, apply a functorial cofibrant factorization to those:

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{Y} & \xrightarrow{\tilde{h}} & \tilde{Z} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y & \xrightarrow{h} & Z \end{array}$$

where $\tilde{g}$, $\tilde{h}$ and $\tilde{hg}$ are cofibrant approximations to g, h and hg respectively. Those exist by virtue of Proposition 8.1.23 of [Hi]. Let W be a iΓ-local object, $\hat{W}$ a simplicial resolution of W. To say for example that g is a iΓ-local equivalence would mean:

$$\underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{Y}, \hat{W}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{X}, \hat{W})$$

is an equivalence in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$, where equivalences satisfy the 2-3 property, so if two of $\tilde{g}^* : \underline{\mathrm{Hom}}(\tilde{Y}, \hat{W}) \to \underline{\mathrm{Hom}}(\tilde{X}, \hat{W})$, $\tilde{h}^* : \underline{\mathrm{Hom}}(\tilde{Z}, \hat{W}) \to \underline{\mathrm{Hom}}(\tilde{Y}, \hat{W})$ or $(\tilde{hg})^* : \underline{\mathrm{Hom}}(\tilde{Z}, \hat{W}) \to \underline{\mathrm{Hom}}(\tilde{X}, \hat{W})$ is a weak equivalence, so is the third, which completes the proof. □

In the statement of Theorem 11.3.1, mention is made of a set J, which exists by virtue of Proposition 4.5.1 which we generalize presently. Its proof uses three results of [Hi], two of which make use of a notion of equivalence, and therefore have to be generalized. Their proof needs the following definition, along with two lemmas, Lemma 5.10 and Lemma 5.11 which we state and prove after those two results.

Definition 5.2. A class of maps S in a symmetric monoidal model category $\mathcal{M}$ is said to be an ideal class of maps if for all $f : A \to B$ in S, for all object C of $\mathcal{M}$, $C \otimes A \to C \otimes B$ is also in S.

Observe that the set of iΓ-local equivalences and the set of Γ-local equivalences form ideal classes.

Lemma 5.3. The class of iΓ-local equivalences forms an ideal class.

Proof. Let $f : X \to Y$ be an iΓ-local equivalence, let C be any object. We show $C \otimes f$ is also a iΓ-local equivalence, that is if W is a iΓ-local object, $\underline{\mathrm{Hom}}(\tilde{Y}, \hat{W}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{X}, \hat{W})$ implies $\underline{\mathrm{Hom}}(\tilde{C} \otimes \tilde{Y}, \hat{W}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{C} \otimes \tilde{X}, \hat{W})$. Let Z be any object. We have:

$$\begin{tikzcd}
\mathrm{Hom}(\tilde{Z}, \underline{\mathrm{Hom}}(\tilde{C} \otimes \tilde{Y}, \hat{W}) \arrow[d, "\cong"] \arrow[dddr, dotted, "\simeq"'] & \\
\mathrm{Hom}(\tilde{Z} \otimes \tilde{C} \otimes \tilde{Y}, \hat{W}) \arrow[r, "\cong"] & \mathrm{Hom}(\tilde{Z} \otimes \tilde{C}, \underline{\mathrm{Hom}}(\tilde{Y}, \hat{W})) \arrow[d, "\simeq"] \\
& \mathrm{Hom}(\tilde{Z} \otimes \tilde{C}, \underline{\mathrm{Hom}}(\tilde{X}, \hat{W})) \arrow[d, "\cong"] \\
& \mathrm{Hom}(\tilde{Z}, \underline{\mathrm{Hom}}(\tilde{C} \otimes \tilde{X}, \hat{W}))
\end{tikzcd}$$

and this for any Z shows by Theorem 17.7.7 of [Hi] that we have our desired equivalence, hence $C \otimes f$ is a iΓ-local equivalence. □

Lemma 5.4. The class of Γ-local equivalences forms an ideal class.

Proof. Let $A \to B$ be a Γ-local equivalence, $\tilde{A} \to \tilde{B}$ a fibrant cofibrant approximation to $A \to B$ that is a cofibration. For any C, $\tilde{C} \otimes \tilde{A} \to \tilde{C} \otimes \tilde{B}$ is a trivial cofibration since $\otimes$ is a Quillen bifunctor on $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ and $\tilde{C}$ is cofibrant. In particular this map is a Γ-local equivalence, so for W a Γ-local object $\mathrm{Hom}(\widetilde{C \otimes B}, \hat{W}) \xrightarrow{\simeq} \mathrm{Hom}(\widetilde{C \otimes A}, \hat{W})$, so that $C \otimes A \to C \otimes B$ is a Γ-local equivalence. □

Definition 5.5. A class of objects $\mathcal{C}$ in a symmetric monoidal model category $\mathcal{M}$ is said to be an ideal class of objects if for any $C \in \mathcal{C}$, for any object X of $\mathcal{M}$, $X \otimes C$ is again in $\mathcal{C}$.

We need the following fact for having an internal right Bousfield localization:

Lemma 5.6. In a right proper, cellular model category $\mathcal{M}$, K a class of objects in $\mathcal{M}$, $\mathcal{K}$ the class of iK-colocal equivalences (Definition 8.2), then the class of i$\mathcal{K}$-colocal objects (Definition 8.1) is an ideal class.

Proof. Let W be a i$\mathcal{K}$-colocal object, $f : X \to Y$ an element of $\mathcal{K}$. We have an equivalence of iLhfC's: $\underline{\mathrm{Hom}}(\tilde{W}, \hat{X}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{W}, \hat{Y})$. Now let $D \in \mathcal{M}$. We wish to show $\underline{\mathrm{Hom}}(\widetilde{D \otimes W}, \hat{X}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\widetilde{D \otimes W}, \hat{Y})$. It suffices to consider for all Z cofibrant in $\mathcal{M}$:

$$\begin{tikzcd}
\mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\widetilde{D \otimes W}, \hat{X})) \arrow[r, dotted] \arrow[d, equal] & \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\widetilde{D \otimes W}, \hat{Y})) \arrow[d, equal] \\
\mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\mathrm{diag}\tilde{D} \otimes \tilde{W}, \hat{X})) \arrow[d, "\cong"] & \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\mathrm{diag}\tilde{D} \otimes \tilde{W}, \hat{Y})) \arrow[d, "\cong"] \\
\mathrm{diag}\mathrm{Hom}(Z \otimes \tilde{D}, \underline{\mathrm{Hom}}(\tilde{W}, \hat{X})) \arrow[r, "\simeq"'] & \mathrm{diag}\mathrm{Hom}(Z \otimes \tilde{D}, \underline{\mathrm{Hom}}(\tilde{W}, \hat{Y}))
\end{tikzcd}$$

having the bottom equivalence by definition of a i$\mathcal{K}$-colocal object, it follows that we have the top horizontal map to be an equivalence, hence by Theorem 17.7.7 of [Hi] we have our desired equivalence, hence $D \otimes W$ is i$\mathcal{K}$-colocal. $\square$

In the above proof, we used:

Lemma 5.7. $\mathrm{diag}\mathrm{Hom}(Z \otimes \tilde{D}, \underline{\mathrm{Hom}}(\tilde{W}, \hat{Y})) \cong \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\mathrm{diag}(\tilde{D} \otimes \tilde{W}), \hat{Y}))$

Proof. We check this componentwise:

$$\begin{aligned}
\mathrm{diag}\mathrm{Hom}(Z \otimes \tilde{D}, \underline{\mathrm{Hom}}(\tilde{W}, \hat{Y}))_n &= \mathrm{Hom}((Z \otimes \tilde{D})_n, (\underline{\mathrm{Hom}}(\tilde{W}, \hat{Y}))_n) \\
&= \mathrm{Hom}(Z \otimes \tilde{D}_n, \underline{\mathrm{Hom}}(\tilde{W}_n, \hat{Y})) \\
&\cong \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\tilde{D}_n \otimes \tilde{W}_n, \hat{Y})) \\
&= \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\mathrm{diag}(\tilde{D} \otimes \tilde{W})_n, \hat{Y})) \\
&= \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\mathrm{diag}(\tilde{D} \otimes \tilde{W}), \hat{Y})_n) \\
&= \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(\mathrm{diag}(\tilde{D} \otimes \tilde{W}), \hat{Y}))_n
\end{aligned}$$

$\square$

The first result is the following:

Proposition[4.5.6 mod] 5.8. If Γ is a set of maps in the left proper, cellular model category $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, if $p : X \to Y$ has the right lifting property with respect to those inclusions of subcomplexes $i : C \to D$ that are iΓ-local equivalences and for which the size of D is at most γ, the cardinal described in the previous section, then p has the right lifting property with respect to all inclusions of subcomplexes that are iΓ-local equivalences.

Proof. Suppose $p : X \to Y$ is a map that satisfies the conditions of the proposition. Let $i : C \to D$ be an inclusion of cell subcomplexes that is a iΓ-local equivalence and such that the size of D is at most γ. It is also a Γ-local equivalence by Lemma 5.11. Since i is a Γ-local equivalence, p has the right lifting property with respect to all inclusions of subcomplexes that are also Γ-local equivalences and for which the size of D is at most γ. The original Proposition 4.5.6 can then be used to conclude p has the right lifting property with respect to all inclusions of subcomplexes that are also Γ-local equivalences, which are also iΓ-local equivalences by Lemma 5.11. This completes the proof. □

The second result is the following:

Lemma[4.5.2 mod] 5.9. If Γ is a set of maps in the left proper, cellular model category $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, if $p : E \to B$ is a fibration with the right lifting property with respect to all inclusions of cell complexes that are iΓ-local equivalences, then it has the right lifting property with respect to all cofibrations that are iΓ-local equivalences.

Proof. The reasoning is similar to the previous result; since iΓ-local equivalences are also Γ-local equivalences, the original Lemma 4.5.2 applies, p has the right lifting property with respect to all cofibrations that are also Γ-local equivalences, which are also iΓ-local equivalences. □

Lemma 5.10. The class of iΓ-local objects of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ coincides with the class of Γ-local objects.

Proof. Let W be a iΓ-local object in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, let $A \to B$ be an object of Γ. Then $\underline{\mathrm{Hom}}(\tilde{B}, \hat{W}) \to \underline{\mathrm{Hom}}(\tilde{A}, \hat{W})$ is an equivalence, which implies the equivalence $\mathrm{Hom}(\tilde{B}, \hat{W}) \to \mathrm{Hom}(\tilde{A}, \hat{W})$, which exactly means that W is Γ-local as well. Conversely, let W be Γ-local, let $A \to B$ in $\Gamma \subset \Gamma - loc.equ's$. Now those form an ideal class, so for any C, $C \otimes A \to C \otimes B$ is a Γ-local equivalence, hence we have an equivalence at the bottom of the commutative diagram below:

$$\begin{tikzcd}
\mathrm{Hom}(\tilde{C}, \underline{\mathrm{Hom}}(\tilde{B}, \hat{W})) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\tilde{C}, \underline{\mathrm{Hom}}(\tilde{A}, \hat{W})) \arrow[d, "\cong"] \\
\mathrm{Hom}(\tilde{C} \otimes \tilde{B}, \hat{W}) \arrow[r, dotted, "\simeq"'] & \mathrm{Hom}(\tilde{C} \otimes \tilde{A}, \hat{W})
\end{tikzcd}$$

and this for any C, so by Theorem 17.7.7 of [Hi], it follows that $\underline{\mathrm{Hom}}(\tilde{B}, \hat{W}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{A}, \hat{W})$, that is W is iΓ-local. □

Lemma 5.11. The set of Γ-local equivalences equals the set of iΓ-local equivalences.

Proof. Let $C \to D$ be a Γ-local equivalence. If A is an object of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $A \otimes C \to A \otimes D$ is also a Γ-local equivalence since those form an ideal class. This means for all W iΓ-local, in particular Γ-local by the preceding lemma:

$$\begin{tikzcd}
\mathrm{Hom}(\widetilde{A \otimes D}, \hat{W}) \arrow[r, "\simeq"] \arrow[d, "\cong"'] & \mathrm{Hom}(\widetilde{A \otimes C}, \hat{W}) \arrow[d, "\cong"] \\
\mathrm{Hom}(\tilde{A}, \underline{\mathrm{Hom}}(\tilde{D}, \hat{W})) \arrow[r, dotted, "\simeq"'] & \mathrm{Hom}(\tilde{A}, \underline{\mathrm{Hom}}(\tilde{C}, \hat{W}))
\end{tikzcd}$$

having the equivalence at the bottom of this diagram implies that we have an equivalence $\underline{\mathrm{Hom}}(\tilde{D}, \hat{W}) \to \underline{\mathrm{Hom}}(\tilde{C}, \hat{W})$ for all iΓ-local object W by Theorem 17.7.7 of [Hi] so $C \to D$ is also a iΓ-local equivalence. Conversely, since equivalences of ihfC's imply equivalences of hfC's, it follows iΓ-local equivalences are also Γ-local equivalences, which completes the proof. □

Proposition[4.5.1 mod] 5.12. In the left proper, cellular model category $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, if Γ is a set of maps, then there exists a set J_Γ of inclusions of cell complexes such that the class of iJ_Γ-cofibrations equals the class of cofibrations that are iΓ-local equivalences.

Proof. The proof is almost identical to that of [Hi], save that it is generalized. We let J_Γ be a set of representative of isomorphism classes of inclusions of cell complexes that are iΓ-local equivalences of size at most γ. This cardinal is the one described in the previous section. The result follows, as in [Hi] in the classical case, from Proposition 5.8 and Lemma 5.9, as well as Corollary 10.5.22 of [Hi], which we use verbatim since it does not make use of a notion of equivalence. □

6 Internal left Bousfield localization

Definition 6.1. For Γ a class of maps in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, $W \in \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is said to be iΓ-local if it is fibrant and for any $f : A \to B$ in Γ, the induced map of iRhfC's $f^* : \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{B}, \hat{W}) \to \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{A}, \hat{W})$ is a weak equivalence in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$.

Definition 6.2. A map $g : X \to Y$ in $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ is said to be a iΓ-local equivalence if for all iΓ-local object W, the induced map of iRhfC's $g^* : \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{Y}, \hat{W}) \to \underline{\mathrm{Hom}}_{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}(\tilde{X}, \hat{W})$ is a weak equivalence in $(\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$.

Theorem 6.3. If Γ is a class of maps in the proper, cellular model category $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, then if we define a class of equivalences on $\underline{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}$ (the category underlying $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$) as being i$\Gamma$-local equivalences, if we define cofibrations as being those of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$, and if define fibrations as being those maps having the right lifting property with respect to those maps that are cofibrations and iΓ-local equivalences, this defines a model structure on $\underline{\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}}$ that we denote by $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}} = \mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}$ and which we call an internal left Bousfield localization of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$ along Γ. Further, $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}\wedge}$ is a left proper, cellular model category.

Proof. We adapt the classical proof, which uses Theorem 11.3.1 of [Hi], which we use as is. The reader is referred to [Hi] for its statement. By Proposition 5.1, the class of iΓ-local equivalences satisfies the 2-3 property. This is one needed assumption of Theorem 11.3.1. Another assumption about this class we need for Theorem 11.3.1 to hold is that it be closed under retracts. This follows in the classical case from Proposition 3.2.4 of [Hi], which holds in the internal case as well. Consider now the set J_Γ provided by Proposition 5.12. Let I be the set of generating cofibrations of $\mathcal{D}_0^{\mathcal{C}^{\mathrm{op}}}$. By definition, I permits the small object argument, hence condition 1 of Theorem 11.3.1 is satisfied. Since every element of J_Γ has a cofibrant domain, small relative to the subcategory of cofibrations by Theorem 12.4.3 of [Hi] as argued in the same paper, hence in particular small relative to J_Γ, this latter satisfies condition 1 of Theorem 11.3.1 as well. Indeed elements of J_Γ are relative I-cell complexes, which are in $I - cof$ by Proposition 10.5.10 of [Hi], and this is the subcategory cof of cofibrations, so $J_\Gamma \subset cof$. Condition 2 of Theorem 11.3.1 is that $J_\Gamma - cof \subset I - cof \cap W$, with $W = i\Gamma - loc.equ's$, and this follows from Proposition 5.12: $J_\Gamma - cof = cof \cap W = I - cof \cap W$. Condition 3 states $I - inj \subset J_\Gamma - inj \cap W$. Proposition 5.12 implies $J_\Gamma - cof$ is a subcategory of $cof = I - cof$, hence $J_\Gamma - inf \supset I - inj$. Finally $I - inj = triv.fibr.$, in particular weak equivalences, which are iΓ-local equivalences, i.e. in W, so

$I - inj \subset J_\Gamma - inf \cap W$. The last condition of Theorem 11.3.1, condition 4a, is satisfied by Proposition 5.12, as argued in [Hi]. This completes the proof that we have a model structure. To show it yields a left proper, cellular model category, we follow exactly the proof of [Hi]. There is no manifest modification going to the general case, so we just refer the reader to the proof in [Hi]. □

7 Results needed in the proof of Thm 5.1.1

We will prove a modified version of Theorem 5.1.1 of [Hi], which necessitates that the model category we are localizing be right proper. For us that would be $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{op}}$, which is left proper, but not necessarily right proper, even if $\mathcal{D}_0^{\mathcal{C}^{op}}$ itself is right proper. For the sake of considering covers, we consider only those model categories $L_\Gamma \mathcal{D}_0^{\mathcal{C}^{op}} = \mathcal{D}_0^{\mathcal{C}^{op}\,\wedge}$ that are right proper as well. Since this may be a strong constraint, we will develop the notion of internal right Bousfield localization not from $\mathcal{D}_0^{\mathcal{C}^{op}\,\wedge}$, but from a generic right proper, cellular model category $\mathcal{M}$.

Proposition[3.2.4 mod] 7.1. If $\mathcal{K}$ is a class of maps in a model category $\mathcal{M}$, the class of i$\mathcal{K}$-colocal equivalences is closed under retracts.

Proof. Let $g : X \to Y$ a i$\mathcal{K}$-colocal equivalence, $f : V \to W$ a retract of g, $\hat{g} : \hat{X} \to \hat{Y}$ and $\hat{f} : \hat{V} \to \hat{W}$ fibrant approximations to g and f respectively such that $\hat{f}$ is a retract of $\hat{g}$. Let C be a i$\mathcal{K}$-colocal object. We want $\underline{\mathrm{Hom}}(\tilde{C}, \hat{V}) \to \underline{\mathrm{Hom}}(\tilde{C}, \hat{W})$ to also be an equivalence. Let $X \in \mathcal{M}$, with cosimplicial resolution $\tilde{X}$. Then consider:

$$\begin{tikzcd}
\mathrm{Hom}(\widetilde{X \otimes C}, \hat{V}) \arrow[rr, "\simeq"] \arrow[d, equal] & & \mathrm{Hom}(\widetilde{X \otimes C}, \hat{W}) \arrow[d, equal] \\
\mathrm{Hom}(\mathrm{diag}\tilde{X} \otimes \tilde{C}, \hat{V}) \arrow[d, "\cong"'] & & \mathrm{Hom}(\mathrm{diag}\tilde{X} \otimes \tilde{C}, \hat{W}) \arrow[d, "\cong"] \\
\mathrm{diag}\mathrm{Hom}(\tilde{X}, \underline{\mathrm{Hom}}(\tilde{C}, \hat{V})) \arrow[rr, dotted] & & \mathrm{diag}\mathrm{Hom}(\tilde{X}, \underline{\mathrm{Hom}}(\tilde{C}, \hat{W}))
\end{tikzcd}$$

i$\mathcal{K}$-colocal objects form an ideal class, so $X \otimes C$ is again i$\mathcal{K}$-colocal, hence $\mathcal{K}$-colocal by Lemma 7.4. Also, g i$\mathcal{K}$-colocal equivalence is a $\mathcal{K}$-colocal equivalence by Proposition 7.5, f is a retract of g, so $\mathcal{K}$-colocal equivalence by the original Proposition of [Hi], so the top map is an equivalence. Hence we have an equivalence at the bottom of the above commutative diagram, and Theorem 17.7.7 of [Hi] allows us to conclude $\underline{\mathrm{Hom}}(\tilde{C}, \hat{V}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{C}, \hat{W})$, hence i$\mathcal{K}$-colocal equivalences are closed under retracts. □

To prove the lifting argument in $R_{\mathcal{K}}\mathcal{M}$, we need to show that a trivial cofibration in $R_{\mathcal{K}}\mathcal{M}$ is also a trivial cofibration in $\mathcal{M}$. The proof of this claim follows from the fact that weak equivalences are also i$\mathcal{K}$-colocal equivalences, which we now prove:

Proposition 7.2. If $\mathcal{K}$ is a class of maps in $\mathcal{M}$, then weak equivalences in $\mathcal{M}$ are also i$\mathcal{K}$-colocal equivalences.

Proof. This follows readily from Theorem 3.1.2 using diagonal objects. □

With this, we can now prove:

Lemma [5.3.2 mod] 7.3. Trivial cofibrations in $R_{\mathcal{K}}\mathcal{M}$ are also trivial cofibrations in $\mathcal{M}$.

Proof. This is verbatim the proof of the original claim in [Hi], save that we use the internal results Proposition 5.1 and Proposition 7.1. □

In the proof of the factorization axiom for model categories, we lift the factorization in the classical case to the internal one by invoking Lemma 7.5 below, which itself follows from:

Lemma 7.4. If $\mathcal{K}$ is a class of maps in $\mathcal{M}$, then i$\mathcal{K}$-colocal objects are also $\mathcal{K}$-colocal objects.

Proof. This follows from Proposition 3.1.1, since $f : X \to Y$ is an equivalence implies that for any W $\underline{\mathrm{Hom}}(\tilde{W}, \hat{X}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\tilde{W}, \hat{Y})$, in particular true for W i$\mathcal{K}$-colocal, hence $X \to Y$ is a i$\mathcal{K}$-colocal equivalence. □

Proposition 7.5. If $\mathcal{K}$ is a class of maps in $\mathcal{M}$, then $\mathcal{K}$-colocal equivalences coincide with i$\mathcal{K}$-colocal equivalences.

Proof. By Proposition 3.1.1, i$\mathcal{K}$-colocal equivalences are also $\mathcal{K}$-colocal equivalences. Conversely, if $g : X \to Y$ is a $\mathcal{K}$-colocal equivalence, for any i$\mathcal{K}$-colocal object W, for any $C \in \mathcal{M}$, we can write:

$$\begin{tikzcd}
\mathrm{Hom}(\widetilde{C \otimes W}, \hat{X}) \arrow[rr, "\simeq"] \arrow[d, "\cong"'] & & \mathrm{Hom}(\widetilde{C \otimes W}, \hat{Y}) \arrow[d, "\cong"] \\
\mathrm{diagHom}(\tilde{C}, \underline{\mathrm{Hom}}(\tilde{W}, \hat{X})) \arrow[rr, dotted, "\simeq"'] & & \mathrm{diagHom}(\tilde{C}, \underline{\mathrm{Hom}}(\tilde{W}, \hat{Y}))
\end{tikzcd}$$

since i$\mathcal{K}$-colocal objects form an ideal class, $C \otimes W$ is also i$\mathcal{K}$-colocal, hence $\mathcal{K}$-colocal by the previous result, so we have the top horizontal equivalence above, hence the one at the bottom as well, and we conclude by using Theorem 17.7.7 of [Hi]. $\square$

8 Internal right Bousfield Localization

Definition 8.1. For $\mathcal{M}$ a model category, $\mathcal{K}$ a class of maps in $\mathcal{M}$, an object W of $\mathcal{M}$ is i$\mathcal{K}$-colocal if cofibrant and if for any $f : A \to B$ in $\mathcal{K}$, we have an induced equivalence in $(\mathcal{M})^{\Delta^{\mathrm{op}}}$ of iLhfC's: $f_* : \underline{\mathrm{Hom}}_{\mathcal{M}}(\tilde{W}, \hat{A}) \to \underline{\mathrm{Hom}}_{\mathcal{M}}(\tilde{W}, \hat{B})$, $\tilde{W}$ a cosimplicial resolution of W and $\hat{A}$ a fibrant approximation to A.

Definition 8.2. A map $g : X \to Y$ in $\mathcal{M}$ is a i$\mathcal{K}$-colocal equivalence if for any $\mathcal{K}$-colocal object W, we have an equivalence in $(\mathcal{M})^{\Delta^{\mathrm{op}}}$ of iLhfC's: $g_* : \underline{\mathrm{Hom}}_{\mathcal{M}}(\tilde{W}, \hat{X}) \to \underline{\mathrm{Hom}}_{\mathcal{M}}(\tilde{W}, \hat{Y})$

The proof of Theorem 5.1.1 makes use of a further notion, which we generalize to our setting:

Definition 8.3. For K a class of objects in $\mathcal{M}$, a map $f : A \to B$ in $\mathcal{M}$ is said to be a iK-colocal equivalence if for any object $k \in K$, we have $\underline{\mathrm{Hom}}_{\mathcal{M}}(\tilde{k}, \hat{A}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{M}}(\tilde{k}, \hat{B})$.

We now state the Theorem, much like Theorem 5.1.1 of [Hi], which states the existence of a internal right Bousfield localization.

Theorem[5.1.1 mod] 8.4. If $\mathcal{M}$ is a right proper, cellular model category, K a set of objects of $\mathcal{M}$, $\mathcal{K}$ the class of iK-colocal equivalences, then the internal right Bousfield localization of $\mathcal{M}$ is a model category structure on $\underline{\mathcal{M}}$ with i$\mathcal{K}$-colocal equivalences as weak equivalences, the same fibrations as $\mathcal{M}$, and for cofibrations those maps that have the left lifting property with respect to those fibrations that are also i$\mathcal{K}$-colocal equivalences.

Proof. The proof, as in [Hi], consists in making sure the axioms of a model category are met. We will only focus on those claims that are different from the proof of [Hi]. The 2-3 property is satisfied because of the dual of Proposition 5.1. The retract argument follows from Lemma 7.2.8 of [Hi], which we use as is, and Proposition 7.1, following the same argument as in [Hi]. The lifting argument involving a cofibration is immediate, since a cofibration in $R_{\mathcal{K}}\mathcal{M}$ has the left lifting property with respect to fibrations that are also i$\mathcal{K}$-local equivalences, i.e. trivial fibrations in $R_{\mathcal{K}}\mathcal{M}$. For the lifting argument involving a trivial cofibration, this follows from Lemma 7.3. Finally for the functorial factorization axiom, we show any map $g : X \to Y$ can be factored as $X \xrightarrow{p} W \xrightarrow{q} Y$, where p is a trivial cofibration in $R_{\mathcal{K}}\mathcal{M}$, p is a fibration in $R_{\mathcal{K}}\mathcal{M}$. This follows readily from the classical case by invoking Proposition 7.5. The same is true of the factorization where now p would be a cofibration, and p a trivial fibration. This completes the proof. $\square$

References

[B] C. Barwick, *On Left and Right Model Categories and Left and Right Bousfield Localizations*, Homology, Homotopy and Applications, vol.**12(2)** (2010), 245-320.

[GK] M. Ghazel and F. Kadhi, *Reedy Diagrams in Symmetric Monoidal Model Categories*, arXiv:1609.09623v1 [math.AT].

[GR] J. J. Gutierrez and C. Roitzheim, *Bousfield Localisations along Quillen Bifunctors*, Applied Categorical Structures, **25** (2017), no.6, 1113-1136.

[Hi] P.S. Hirschhorn, *Model Categories and their Localizations*, Math. Surveys and monographs Series, Vol. **99**, AMS, Providence, 2003.

[Ho] M. Hovey, *Model Categories*, Mathematical Surveys and monographs, Vol. **63**, Amer. Math. Soc., Providence, 1998.

[K] Kelly, *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Note Series, **64**, Cambridge University Press, New York, 1982.

[McL] S. Mac Lane, *Categories for the Working Mathematician*, 2nd Ed., Springer Verlag, New York, 2010.

[TV] B.Toen, G.Vezzosi, *Homotopical Algebraic Geometry I: Topos Theory*, Adv. Math. **193**(2005), no.2, 257-372

[WY] D. White and D. Yau, *Right Bousfield Localization and Operadic Algebras*, arXiv:1512.07570v2 [math.AT].

[W] D. White, *Monoidal Bousfield Localizations and Algebras over Operads*, arXiv:1404.5197v2 [math.AT]

Models of Discrete Conformal Geometry

William E. Wood
University of Northern Iowa

Conformal geometry is the study of mappings that preserve angles. The classic examples come from elementary complex analysis: one-to-one analytic mappings between two regions of the complex plane, including common functions like polynomials and exponentials (with domains appropriately restricted). *Discrete* conformal geometry refers to a theory moving results of classical conformal geometry and complex analysis to the discrete setting of graphs.

To do this, we must develop some notion of what discrete conformal mapping should be. We would like to mimic the classical situation as best we can, so let's summarize some features that characterize them:

Angles are preserved. This is our definition, but what could this mean on a graph?

Infinitesimal circles are mapped to infinitesimal circles. This gets a notion of angle in a more useful way. The Cauchy-Riemann Equations from complex analysis say that an analytic mapping locally behaves like a rotation and a dilation, maps which take circles to circles.

Extremal length is preserved. Extremal length is a powerful conformal invariant that translates particularly well to the discrete setting. We will define it formally in Section 2.

Mappings provide a conformal coordinate system on a region. If we have a coordinate system in a region where, for example, we might have a grid of perpendicular axes, then mapping the region conformally should carry the grid to another one with coordinate curves remaining perpendicular. We can think of these coordinates as instructions on how to conformally "straighten" the set.

1 Circle Packing

As we look to discretize these properties, we recognize that "preservation of angles" is a non-starter; we have no notion of angle for a graph. The property that infinitesimal circles map to infinitesimal circles, however, can be made to work.

This gives rise to the idea of a circle packing, which is a collection of circles that meet in externally tangent triples. There is a natural graph associated to

10.5281/zenodo.2579477

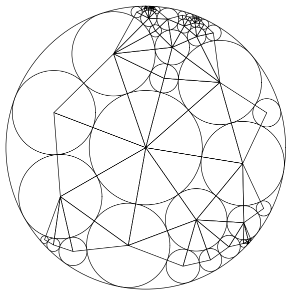


Figure 1: *A circle packing with centers connected to show the tangency graph. All boundary circles are internally tangent to the unit circle, illustrating the discrete Riemann mapping theorem. Drawn with Ken Stephenson's* CirclePack *software.[11]*

any circle packing. Vertices correspond to the circles and two vertices form an edge if their corresponding circles are tangent in the packing. We can see this graph by connecting the centers of the packed circles.

The radii of the circles are positive weights on the vertices. If we change the radii of the circles without changing the tangencies, the packing will look different but the underlying graph is unchanged. This is our notion of a discrete analytic function, borrowing from our working property of conformal mapping but removing the discrete-unfriendly word "infinitesimal." We should expect to see this "infinitesimal" emerge as some kind of limit in our discrete model. We will illustrate exactly that by first considering one of the most celebrated theorems in classical analysis.

Theorem 1 (Riemann Mapping Theorem) *Any open simply connected proper subset of the complex plane can be mapped to the open unit disk by a bijective analytic map. This map is unique up to automorphisms of the disk.*

This is amazing. Conformality seems to be a strong condition, but it is sufficient to take any pathological simply connected monstrosity nicely onto the disk. Moreover, the condition is essentially unique after accounting for the three-parameter family of maps from the disk to itself.

Here is the discrete analog:

Theorem 2 (Discrete Riemann Mapping Theorem) *Any finite triangulation of a disk realizes a circle packing whose boundary circles are internally tangent to the unit circle.*

This theorem was established as a linchpin for discrete conformal geometry by William Thurston (although it was actually proved much earlier, c.f. [2], [8], [13]). To see how the two versions tie to together, consider a bounded simply connect region R in the plane. Cover the plane with the "penny packing" formed by a packing of identical circles with each circle tangent to exactly six neighbors. Carve out a portion of this packing including every circle whose interior contains at least one point in R. If the radius of the packed circles is small relative to R, then the region filled by this collection of circles should approximate R.

This is a circle packing. By the Discrete Riemann Mapping Theorem, the radii of these currently identical circles may be adjusted without altering the underlying combinatorics so that the boundary circles lie internally tangent to the unit circle, and indeed algorithms exist to approximate these radii. We impose uniqueness by, say, forcing some chosen circle to map to the origin and another to lie on the positive real axis. This discrete analytic mapping takes a point in R that is a center of a circle to a point inside the unit disk, which is the center of its corresponding circle. We can extend continuously to points that are not centers by using barycentric coordinates. Now we see how to get that limiting process we wanted. Repeat this procedure on penny packings of smaller and smaller radius (with consistent normalizations), giving better and better approximations to the region R.

Theorem 3 *The sequence of discrete analytic mappings described above converges uniformly on compact sets to the Riemann Mapping of R to the disk, with the corresponding normalizations.*

The first version of this theorem was proved by Rodin and Sullivan in [9] and has since been extended in various ways. He and Schramm [7] proved a version powerful enough to count as an alternate proof of the Riemann Mapping Theorem itself. (We are overlooking a long story about the precise statements of these results, among other things. See [12] and the references therein.)

2 Extremal Length

Of significant interest is determining which sets can be mapped to one another by a conformal mapping, in which case we call the sets *conformally equivalent*. The Riemann Mapping Theorem essentially says that all proper open simply connected subsets of the plane are conformally equivalent. The story gets more interesting with quadrilaterals, where we further require that the mapping to carry vertices to vertices. A *conformal invariant* is a property that must be shared by two conformally equivalent sets.

Define a *topological quadrilateral* to be a simply connected domain with four distinguished vertices that divide the boundary into four arcs. Choose one non-intersecting pair of these arcs to be the "top" and "bottom."

Extremal length is a conformal invariant developed by Lars Ahlfors [1] and has emerged as a powerful tool in conformal geometry. We will skip to the punchline to motivate this tool. The Riemann Mapping Theorem tells us that any open simply connected region in the complex plane can be mapped conformally to a disk. Consider this domain as a topological quadrilateral we wish to map to a rectangle with vertices mapping to corresponding vertices. The Riemann mapping is unique up to three parameters of automorphisms, so specifying four points costs us a degree of freedom. We pay this debt in the aspect ratio (length divided by width) of the image rectangle, which is forced upon us and is a conformal invariant of the quadrilateral. This will turn out to be the extremal length, although we actually define it in terms of path families.

Define a *metric* to be a positive function on the quadrilateral Q and its *area* to be $\int\int_Q m^2\ dA$. We restrict ourselves to the set Λ of metrics with positive area. If γ is a curve lying in Q, its length with respect to a metric m is $\int_\gamma m\ d\gamma$. Finally, let Γ be the set of all rectifiable curves in Q connecting the top arc to the bottom. Extremal length is defined as follows.

$$\mathrm{EL}(Q) = \sup_{m\in\Lambda} \frac{\inf_{\gamma\in\Gamma}(\int_\gamma m\ |dz|)^2}{\int\int m^2\ dA}$$

The infimum in the numerator is finding the shortest (squared) length path connecting opposite sides with respect to a metric. The denominator is the area. Thinking of rectangles, this is just

$$\frac{\text{length}^2}{\text{area}} = \frac{\text{length}^2}{\text{length}\times\text{width}} = \frac{\text{length}}{\text{width}},$$

which is the aspect ratio. It thus makes some sense to define the quantity inside the supremum as the aspect ratio of the metric. The supremum then searches through all metrics on the quadrilateral for the one with the largest aspect ratio. It turns out that this metric exists and is indeed the norm of the derivative of the Riemann mapping onto a rectangle. Note that the ratio of length over width is automatically scale invariant so we often restrict to metrics normalized to area one. To see that this is a conformal invariant, consider another quadrilateral Q' that is conformally equivalent to Q. By definition, there is some conformal mapping ρ from Q to Q', and $|\rho'|$ will necessarily by included in Λ. In other words, the supremum automatically sifts through conformal equivalences. We should note that the definition of extremal length puts no restrictions on the set Γ of curves and there are lots of reasons to study other curve families, but curves connecting opposite sides of quadrilaterals are sufficient for our purposes. We also point out that the choice of which pair of arcs are the top and bottom does matter, but choosing the other pair simply reciprocates the aspect ratio (i.e., switches the roles of length and width).

This definition is easy to port to a discrete setting – we just change the regions to graphs, the curves to vertex paths, and the integrals to sums.

$$\mathrm{EL}(G) = \sup_{\rho\in\hat{\Lambda}} \frac{\inf_{\gamma\in\hat{\Gamma}}(\sum_{v\in\gamma} m(v))^2}{\sum_{v\in V} m(v)^2}$$

G is a combinatorial quadrilateral, defined as a planar triangulation with boundary divided into four vertex paths, with two disjoint arcs designated the

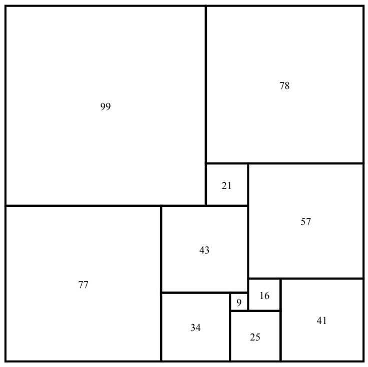


Figure 2: *A square tiling of a* 176×177 *rectangle. The extremal length of the associated discrete quadrilateral is thus* $\frac{177}{176}$. *This is one of the first examples of a* perfect tiling *in which no two squares are congruent [4].*

"top" and "bottom," just as in the classical case. $\hat{\Lambda}$ is the set of positive functions m on the vertices V of G with area $\sum_{v \in V} m(v)^2 > 0$, and $\hat{\Gamma}$ is the set of vertex paths in G connecting the top of G to the bottom.

The proof that extremal length exists is not too hard. Consider a vector space whose basis is the set of vertices of G. Then a metric, being an assignment of values to each vertex, is an element of this finite-dimensional vector space. Since extremal length is scale invariant, we may restrict to unit area metrics, i.e. points on the unit sphere in our vector space. One need only verify that the aspect ratio is a continuous function, so we are looking for a maximum of a continuous function on a compact set, which always exists. Uniqueness can be proved from convexity of the sphere. See [5] for details.

So the metric exists and is unique, but what is it? It turns out that if we use the graph as a tangency graph for squares of side length $m(v)$, then those squares will fit together in a perfect rectangle. Like the classical case, we have no control over the aspect ratio of the rectangle, which will be the extremal length. These square tilings of rectangles are similar to circle packings except that the non-smoothness of squares requires us to allow some degenerate cases.

An interesting schism in the theory now emerges. If we refine the graph as we did with circle packings, the discrete mappings induced by square tilings will not generally converge to the Riemann mapping of a region to a rectangle. Extremal length captures some aspects of conformality and circle packings capture others. Discretization has finally cost us.

3 Electric Networks and Other Models

We now consider how Riemann himself thought about conformal mapping. Suppose we have a topological cylinder made from some conductive metal. Connect a battery to the two boundary circles to draw a current from one boundary circle to the other. The electrons will move between these circles along flow lines while the curves of equal potential will necessarily be orthogonal to these flow lines. But then these flow lines and equipotential curves give conformal coordinates that map the topological cylinder to a straight rectangular one, which was our final feature of conformal mappings. The resistance of this is system – which is clearly predetermined by the physical interpretation – will be captured by the height of the rectangular cylinder. This is extremal length again.

To discretize this idea, consider the edges of a graph to be wires with unit resistance. If we draw a current between the boundary components, then there will be some effective resistance between the components. This can sometimes be computed using Kirchoff's Laws. This approach brings us back to square tiling, although here the squares are represented by edges rather than vertices. See [3] for a nice exposition.

A similar model is random walk. Doyle and Snell [6] show that this is equivalent to the electric network model by showing that they both arise from discrete harmonic functions (i.e., functions whose values are the averages of the values of neighboring points). These functions are uniquely determined by boundary conditions, so the two systems must yield the same functions. The classical analog is Brownian motion.

Other models can be obtained from different kinds of packings. We have studied circles and squares, but many similar results can be obtained by packing more general shapes. We may also relax the tangency condition (necessary if we start varying the shapes of tiles). Most models tend to behave similarly if the underlying triangulations have bounded degree, meaning there is an $N > 0$ such that every vertex has at most N adjacent vertices. This condition can usually translate into a bound on how the discrete function deviates from a conformal mapping, inviting techniques of *quasiconformal analysis* to force nice limiting behaviors. Without this condition, however, quasiconformal methods fail and the peculiarities of the models reveal themselves.

4 Onward

All of our characterizations of conformal mapping have discrete analogs, but different characterizations required meaningfully different models. Discrete conformal geometry has grown into a large field with powerful theoretical and computational results. As mathematicians push these results further, however, the gaps between the different models become more interesting.

For example, let's tweak our definition of extremal length by assigning the metric to the edges of the graph instead of to the vertices. Everything else about the definition as maximal aspect ratio remains the same. It does not seem at first blush that this should change the story much, but it turns out that the switch from vertices to edges recaptures the circle packing versus square tiling dichotomy. See [14].

Many questions remain open about each of these models, and many more are available if we want to look deeply into the differences among them. The insights we gain into how these models do and do not capture classical behaviors have potential to lead to more effective use of these tools as we pursue questions in both settings.

References

[1] L. Ahlfors, *Conformal invariants: Topics in geometric function theory*, McGraw-Hill, Inc., 1973.

[2] E. M. Andreev, *Convex Polyhedra in Lobacevskii Space*, Mat. Sbornik, Volume 81, No. 123, 1970.

[3] B. Bollobás, *Modern Graph Theory*, Springer, 1998.

[4] R. Brooks, C. Smith, A. Stone, and W. Tutte, *The Dissection of Rectangles into Squares*, Duke Math. J. 7, pp. 312-340, 1940.

[5] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Squaring rectangles: the finite Riemann mapping theorem*, Contemp. Math., vol. 169, Amer. Math. Soc., Providence, RI, pp. 133-212, 1994.

[6] P. G. Doyle and J. L. Snell, *Random walks and electric networks*, The Carus Mathematical Monographs, no. 22, Math. Association of America, 1984.

[7] Z.-X. He and O. Schramm, *On the convergence of circle packings to the Riemann map*, Invent. Math., Volume 125, Issue 2, pp 285-305, 1996.

[8] P. Koebe, *Kontaktprobleme der Konformen Abbildung*, Ber. Schs. Akad. Wiss. Leipzig, Math.-Phys. Kl. 88: 141164, 1936.

[9] B. Rodin and D. Sullivan, *The convergence of circle packings to the Riemann mapping*, J. Differential Geom. Volume 26, Number 2, 349-360, 1987.

[10] O. Schramm, *Square tilings with prescribed combinatorics*, Israel Journal of Mathematics, Volume 84, Issue 1-2, pp. 97-118, 1993.

[11] K. Stephenson, *CirclePack*, software, http://www.math.utk.edu/ kens/CirclePack.

[12] K. Stephenson, *Introduction to Circle Packing*, Cambridge University Press, 2005.

[13] W. Thurston, The finite Riemann mapping theorem, Invited talk at the International Symposium at Purdue University on the occasion of the proof of the Bieberbach conjecture, 1985.

[14] W. Wood, *Combinatorial Modulus and Type of Graphs*, Topology and its applications, 159, no. 17, pp. 2747–2761, 2009.

Positive solutions to some systems of nonlinear Diophantine equations

June 09, 2017

Christopher Briggs[1]
Department of Mathematics
Embry-Riddle Aeronautical University

1 Introduction

We noticed that

$$\begin{aligned} 1+5 &= 2*3 \\ 2+3 &= 1*5 \end{aligned}$$

and we wondered if there is another quadruple of positive integers satisfying the system:

$$\begin{aligned} a+b &= cd \\ c+d &= ab. \end{aligned}$$

This led us to consider more generalized sets of Diophantine equations:

$$\begin{aligned} a_{1,1}+a_{1,2} &= a_{2,1}a_{2,2} \\ a_{2,1}+a_{2,2} &= a_{3,1}a_{3,2} \\ &\vdots \\ a_{n-1,1}+a_{n-1,2} &= a_{n,1}a_{n,2} \\ a_{n,1}+a_{n,2} &= a_{1,1}a_{1,2}. \end{aligned} \tag{1}$$

We will see that System 1 has finitely many positive integer solutions for all n. We define a sequence a_n by the number of (distinct) solutions to the system, and consider the properties of the sequence $\{a_n\}$.

[1]This presentation is based on joint work [1] with Yasuyuki Hirano of Naruto University of Education and Hisa Tsutsui of Embry-Riddle Aeronautical University.

10.5281/zenodo.2579463

More generally, we prove that the system

$$\begin{aligned}
\sum_{j=1}^{m} a_{1,j} &= \prod_{j=1}^{m} a_{2,j} \\
\sum_{j=1}^{m} a_{2,j} &= \prod_{j=1}^{m} a_{3,j} \\
&\vdots \\
\sum_{j=1}^{m} a_{n-1,j} &= \prod_{j=1}^{m} a_{n,j} \\
\sum_{j=1}^{m} a_{n,j} &= \prod_{j=1}^{m} a_{1,j}
\end{aligned} \tag{2}$$

has finitely many solutions over the positive integers.

2 Finiteness of terms

We begin with a result on solutions to Equations (2) over $\mathbb{C}$, then narrow our focus to positive solutions to Equations (1). As a first step to showing that the number of positive solutions to (2) is finite, we find an upper bound on the smallest-normed element of any solution to (2).

Proposition 1. *For any solution to Equations (2) over $\mathbb{C}$ we have $\min_{i,j}\{|a_{i,j}|\} \leq m^{\frac{1}{m-1}}$, with equality if and only if $|a_{i,j}| = m^{\frac{1}{m-1}}$ for all i, j.*

Proof. We assume $\min_{i,j}\{|a_{i,j}|\} \geq m^{\frac{1}{m-1}}$ and show equality. Under the hypotheses, for each i, $|\prod_{j=1}^{m} a_{i,j}| \geq (\min_j\{|a_{i,j}|\})^{m-1} \max_j\{|a_{i,j}|\} \geq (m^{\frac{1}{m-1}})^{m-1} \max_j\{|a_{i,j}|\} = m \max_j\{|a_{i,j}|\}$. We have

$$\begin{aligned}
\Big|\sum_{j=1}^{m} a_{1,j}\Big| &= \Big|\prod_{j=1}^{m} a_{2,j}\Big| \geq m \max_j\{|a_{2,j}|\} \geq \sum_{j=1}^{m} |a_{2,j}| \geq \Big|\sum_{j=1}^{m} a_{2,j}\Big| \\
\Big|\sum_{j=1}^{m} a_{2,j}\Big| &= \Big|\prod_{j=1}^{m} a_{3,j}\Big| \geq m \max_j\{|a_{3,j}|\} \geq \sum_{j=1}^{m} |a_{3,j}| \geq \Big|\sum_{j=1}^{m} a_{3,j}\Big| \\
&\vdots \\
\Big|\sum_{j=1}^{m} a_{n-1,j}\Big| &= \Big|\prod_{j=1}^{m} a_{n,j}\Big| \geq m \max_j\{|a_{n,j}|\} \geq \sum_{j=1}^{m} |a_{n,j}| \geq \Big|\sum_{j=1}^{m} a_{n,j}\Big| \\
\Big|\sum_{j=1}^{m} a_{n,j}\Big| &= \Big|\prod_{j=1}^{m} a_{1,j}\Big| \geq m \max_j\{|a_{1,j}|\} \geq \sum_{j=1}^{m} |a_{1,j}| \geq \Big|\sum_{j=1}^{m} a_{1,j}\Big|
\end{aligned}$$

Hence all terms above are equal, and $m \max_j\{|a_{i,j}|\} = \sum_{j=1}^m \{|a_{i,j}|\}$, implying $|a_{i,j}| = \max_{k,l}\{|a_{k,l}|\}$ for all i, j. Let $a = \max_{k,l}\{|a_{k,l}|\}$. We have $ma = a^m$, so $a = m^{\frac{1}{m-1}}$. $\square$

The proof that the terms of A275234 are finite relies on the insight that, from each row to the next on the left side of the equations in the system (2), the sum can increase by at most one. The following lemma establishes this fact.

Lemma 2. *If $m \geq 2$, $a_i \in \mathbb{N}$ for $1 \leq i \leq m$, and $a_1 \leq \cdots \leq a_m$, then $\sum_{j=1}^m a_j \leq m - 1 + \prod_{j=1}^m a_j$, with equality if and only if $a_{m-1} = 1$. In the case of equality, we have $\sum_{j=1}^m a_j = m - 1 + a_m$.*

Proof. Define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(x_1, \ldots, x_m) = \prod_{j=1}^m x_j - \sum_{j=1}^m x_j$. We have $g(1, \ldots, 1) = 1 - m$. Also, $\frac{\partial g}{\partial x_k}(r_1, \ldots, r_m) > 0$ whenever $r_j \geq 1$ for each j and $r_l > 1$ for some $l \neq k$. In particular, $\frac{\partial g}{\partial x_k}(r_1, \ldots, r_m) > 0$ if $r_j \geq 1$ for each j and $r_j > 1$ for at least two distinct j. Thus $g(a_1, \ldots, a_m) > 1 - m$ if $a_{m-1} > 1$. For the converse, if $a_{m-1} = 1$, then $\sum_{j=1}^m a_j = m - 1 + a_m = m - 1 + \prod_{j=1}^m a_j$. $\square$

Since the sums in the rows of (2) can increase by at most one from one row to the next, from no one row to the next can the decrease be as great as the number of rows in the system. The following two results show that, given this restriction, for each i there are finitely many choices for $a_{i,1}, \ldots, a_{i,m}$.

Lemma 3. *Fix $m \geq 2$ and $B > 0$. Let $a_i \in \mathbb{N}$ for $1 \leq i \leq m$ and $a_1 \leq \cdots \leq a_m$. If $\prod_{j=1}^m a_j$ is sufficiently large (depending on B), then either $\sum_{j=1}^m a_j = a_m + m - 1$ or $B + \sum_{j=1}^m a_j < \prod_{j=1}^m a_j$.*

Proof. Assume $\sum_{j=1}^m a_j \neq a_m + m - 1$. By Lemma 2, $a_{m-1} \geq 2$. Let g be as in the proof of Lemma 2. We have $\frac{\partial g}{\partial x_j}(a_1, \ldots, a_m) \geq 1$, so

$$g(a_1, \ldots, a_m) \geq g(1, \ldots, 1, 2, 2) + (a_m - 2) + (a_{m-1} - 2) + \sum_{j=1}^{m-2}(a_j - 1).$$

Reorganizing terms, and noting $g(1,\dots,1,2,2) = 4 - (1+\dots 1+2+2) = 2-m$, we have

$$\begin{aligned}
\prod_{j=1}^{m} a_j - \sum_{j=1}^{m} a_j &\geq (2-m) + (a_m - 2) + (a_{m-1} - 2) + \sum_{j=1}^{m-2}(a_j - 1) \\
\prod_{j=1}^{m} a_j - \sum_{j=1}^{m} a_j &\geq -2 - m + a_m + a_{m-1} + \sum_{j=1}^{m-2}(a_j) + \sum_{j=1}^{m-2}(-1) \\
\prod_{j=1}^{m} a_j - \sum_{j=1}^{m} a_j &\geq -2 - m + a_m + a_{m-1} + \sum_{j=1}^{m-2}(a_j) + 2 - m \\
\prod_{j=1}^{m} a_j &\geq -2m + 2\sum_{j=1}^{m} a_j \\
\sum_{j=1}^{m} a_j &\leq m + \frac{1}{2}\prod_{j=1}^{m} a_j.
\end{aligned}$$

We want $m + \frac{1}{2}\prod_{j=1}^{m} a_j < -B + \prod_{j=1}^{m} a_j$, which is true if $\prod_{j=1}^{m} a_j > 2(m+B)$. □

Corollary 4. *If $m \geq 2$ and $B > 0$ then $B + \sum_{j=1}^{m} a_j < \prod_{j=1}^{m} a_j$ for all but finitely many choices of $a_1, \dots, a_m \in \mathbb{N}$.*

We now unite the preliminary results to prove the main result of the section.

Theorem 5. *There are finitely many positive solutions to Equations (2).*

Proof. Let $\{a_{i,j}\} \subset \mathbb{N}$ satisfy Equations (2). Assume $a_{i,j} \leq a_{i,j+1}$ for each $1 \leq i \leq n$ and $1 \leq j < m$. Solving for 0 in each equation in (2) and summing the results yields

$$\sum_{i=1}^{n}\Big(\sum_{j=1}^{m} a_{i,j} - \prod_{j=1}^{m} a_{i,j}\Big) = 0. \tag{3}$$

By Lemma 2, for each i we have $\sum_{j=1}^{m} a_{i,j} - \prod_{j=1}^{m} a_{i,j} \leq m-1$, so Equation (3) is impossible if $\prod_{j=1}^{m} a_{i,j} - \sum_{j=1}^{m} a_{i,j} > (m-1)(n-1)$ for any i. Then by Corollary 4, for each i there are finitely many choices $a_{i,1}, \dots, a_{i,m}$ with $a_{i,m-1} > 1$. To complete the proof we show that there are finitely many choices with $a_{i,m-1} = 1$.

Suppose $a_{i,m-1} = 1$ and $a_{i+1,m-1} > 1$. Because $a_{i,m} = 1 - m + \prod_{j=1}^{m} a_{i+1,j}$, if $a_{i,m}$ is sufficiently large, then by Lemma 3 we have $\prod_{j=1}^{m} a_{i+1,j} - \sum_{j=1}^{m} a_{i+1,j} > (m-1)(n-1)$. That contradicts Equation (3), so there are finitely many choices for $a_{i,m}$.

By Lemma 2 and Equation (3), $a_{l,m-1} > 1$ for some l. So the final case is $a_{i,m-1} = \dots = a_{i+k-1,m-1} = 1$ and $a_{i+k,m-1} > 1$. We know $a_{i+k-1,m}$ is bounded above. Since $a_{i+k-1,m} = (k-1)(m-1) + a_{i,m}$, the same bound applies to $a_{i,m}$. □

Corollary 6. *The sequence A275234 defined by the number of positive solutions to Equations (1) has finite terms.*

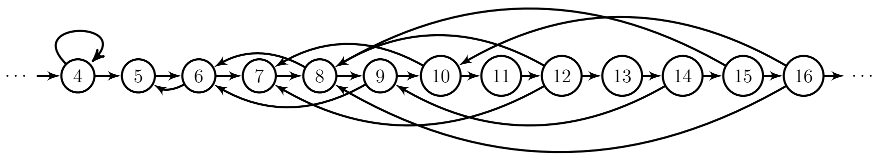


Figure 1: an infinite directed graph

3 Finding the terms

The authors created a Python script which computes the n^{th} term of the subject sequence A275234 (or, at the user's option, a list of the solutions). While the script was created with performance in mind, there is no guarantee of optimum performance. The first several terms are

$$1, 2, 2, 3, 3, 5, 4, 7, 7, 12, 12, 21, 22, 37, 47, 72, 93, 145, 198, 303, 427, \ldots$$

Computation time becomes an issue soon after this point.

We discuss a potential efficient alternate approach to finding larger terms. Consider the infinite directed graph in Figure 1. The nodes are the positive integers, and there is an edge from m to n whenever there exists a pair of positive integers a, b such that $ab = m$ and $a + b = n$. For example, there is an edge from 5 to 6 because $5 \cdot 1 = 5$ and $5 + 1 = 6$. There is an edge from 6 to 5 because $2 \cdot 3 = 6$ and $2 + 3 = 5$. The number of edges originating at a node n is the number of divisors of n which are less than or equal to $\sqrt{n}$ (A038548), and the number of edges terminating at a node n is equal to $\lfloor \frac{n}{2} \rfloor$ (A004526). We call an edge from m to $m - k$ with $k > 0$ a *chute* (of length k) (in analogy with the board game Chutes and Ladders). The number of chutes of length k is equal to the number of divisors of $k + 1$ which are less than or equal to $\sqrt{k+1}$ (A038548).

The significance of the graph is that every closed walk of length n constitutes a solution to Equations (1). For example, with $n = 5$ the closed walk $6 \to 7 \to 8 \to 6 \to 5 \to 6$ corresponds to the solution

$$\begin{aligned} 6 + 1 &= 7 \cdot 1 \\ 7 + 1 &= 2 \cdot 4 \\ 2 + 4 &= 2 \cdot 3 \\ 2 + 3 &= 1 \cdot 5 \\ 1 + 5 &= 1 \cdot 6 \end{aligned}$$

One may consider the infinite adjacency matrix A corresponding to the graph. The matrix is sparse in the sense that all but finitely many entries in each row and column are 0. In particular, the sum of the i^{th} row is the i^{th} term of A038548, and the sum of the j^{th} column is $\lfloor \frac{j}{2} \rfloor$ (A004526). The matrix A^k has as its n^{th} diagonal entry the number of closed walks starting at the node n. In light of the proof of Lemma 3, in computing the number of closed walks of length k it suffices to find the k^{th} power of the upper-left square submatrix of dimension $2k+4$. Unfortunately, the walks counted in this way are not necessarily distinct in the sense of distinct solutions to Equations (1), and so we have only upper bounds on the terms of A275234. The authors remain interested in finding a combinatorial argument which bridges the adjacency matrix and the terms of A275234.

References

[1] C. Briggs, Y. Hirano and H. Tsutsui, Positive Solutions to some Systems of Diophantine Equations, *J. Int. Seq.* **19** (2016), 16.8.4

[2] W. K. Chen, *Graph Theory and its Engineering Applications*, World Scientific Publishing Co. Pte. Ltd., 1997.

[3] C. Coates, Flow-graph solutions of linear algebraic equations, *IRE Trans. Circuit Theory* **6** (1959), 170–187.

[4] S. J. Mason, Feedback theory - some properties of signal flow graphs, *Proc. IRE* **41** (1953), 1144–1156.

[5] S. J. Mason, Feedback theory - further properties of signal flow graphs, *Proc. IRE* **44** (1956), 920–926.

[6] W. H. Mills, A system of quadratic Diophantine equations, *Pac. J. Math.* **3** (1953), 209–220

[7] C. E. Shannon, *The Theory and Design of Linear Differential Equation Machines*, Fire Control of the US National Defense Research Committee: Report 411, Section D-2, 1942.

[8] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

[9] K. Thulasiraman and M. N. S. Swamy, *Graphs: Theory and Algorithms*, John Wiley & Sons Inc., 1992.

MULTIPLICATIVE SETS OF IDEMPOTENTS IN A SEMILOCAL RING

YASUYUKI HIRANO

Department of Mathematics, Naruto University of Education,
Naruto 772-8502, Japan

An element e of a ring R is called an idempotent if $e^2 = e$. An idempotent e is said to be primitive if there are no two non-zero idempotent $f,\ g \in R$ such that $e = f + g$ and $fg = gf = 0$.

Proposition 1. *Let K be a field of charactristic $p \neq 2$. Let R be a K-subalgebra of the ring $M_n(K)$ of $n \times n$ matrices over K containing matrix units $e_{11}, e_{22}, \cdots, e_{nn}$. Let M denote the set consisting of primitive idempotents and 0. Suppose that, for any $e,\ f \in M$, ef is either an idempotent or a nilpotent element. Then R is isomorphic to a K-subalgebra of the ring $T_n(K)$ of all upper triangular matrices over K.*

Proof. Assume that $e_{ij}, e_{ji} \in R$ for some $i \neq j$. Then R contain two primitive idempotents $e = e_{ii} + e_{ij}$ and $f = e_{ii} + e_{ji}$. We see that $ef = 2e_{ii}$. Since $char(K) \neq 2$, $2e_{ii}$ is neither an idempotent nor a nilpotent element. Hence, if $e_{ij} \in R$ for some $i \neq j$, then $e_{ji} \notin R$. Now we define an order on the set $\{1, 2, \cdots, n\}$. If $e_{ij} \in R$, then we define $i \leq j$. Since $e_{ii} \in R$ for all $i \in \{1, 2, \cdots, n\}$, we have $i \leq i$. If $i \leq j$ and $j \leq k$, then $e_{ij},\ e_{jk} \in R$, and hence $e_{ik} = e_{ij}e_{jk} \in R$. Therefore $i \leq k$. If $i \leq j$ and $j \leq i$, then $e_{ij},\ e_{ji} \in R$. As we saw in the first paragraph of the proof, $i = j$ in this case. Therefore $\leq$ is a partial order on $\{1, 2,\ \cdots, n\}$. Let m be a minimal element of the ordered set $\{1, 2,\ \cdots, n\}$. Then $e_{mj} \notin R$ for any $j \neq m$. Renumbering the elements in $\{1, 2,\ \cdots, n\}$, we may assume that $m = 1$. Then we see $R \subset e_{11}K + (e_{22} + \cdots + e_{nn})R(e_{22} + \cdots + e_{nn})$. Using induction on n, $(e_{22} + \cdots + e_{nn})R(e_{22} + \cdots + e_{nn})$ is isomorphic to a K-subalgebra of the ring $T_{n-1}(K)$. Hence R is isomorphic to a K-subalgebra of $T_n(K)$.

Modified version of this article has been submitted elsewhere for publication.

10.5281/zenodo.2579473

The following example show that the proposition above is not true when the field K is of characteristic 2.

Example 1. *Consider the ring $R = M_2(GF(2))$ of 2×2 matrices over the Galois field $GF(2)$ and let M denote the set consisting of all primitive idempotents in R and zero. We can easily see that for any $e,\ f \in M$, ef is either an idempotent or a nilpotent element.*

Next, we prove the following.

Proposition 2. *Let e be a primitive idempotent of a ring R. If ef is a non-zero idempotent of R for some element $f \in R$, then ef is a primitive idempotent.*

Proof. Assume that $ef = a + b$ for some orthogonal idempotents $a,\ b \in R$. Then $a + b = ef = ea + eb$, and so $a = (a + b)a = (ea + eb)a = ea$. Similarly, we have $b = eb$. We can easily see that $e - ae$ and ae are orthogonal idempotents and $e = (e - ae) + ae$. Since e is a primitive idempotent, either $e = ae$ or $ae = 0$ holds. If $e = ae$, then $b = eb = aeb = ab = 0$. On the other hand, if $e = ae$, then $a = a^2 = aea = 0$. This proves that ef is primitive.

Let R be a ring. Let M and E denote the set consisting of all primitive idempotents in R and zero and the set of idempotents in R, respectively. If S is a multiplicatively closed set of idempotents in R containing 0, then $M \cap S$ is also multiplicatively closed.

By Zorn's lemma, we have the following.

Proposition 3. *Every multiplicatively closed subset of M (resp. E) is contained in a maximal multiplicatively closed subset of M (resp. E).*

Example 2. *Let $M_2(K)$ be a ring of 2×2 matrices over a field K. We can see that $(e_{11} + e_{12}K) \cup \{0\}$ is a maximal multiplicatively closed subset of M.*

Theorem 1. *Let R be a ring and let M denote the set consisting of all primitive idempotents in R and zero. Suppose that there are primitive orthogonal idempotents $e_1, e_2, \cdots, e_n$ of R such that $1 = e_1 + e_2 + \cdots + e_n$. Then $\{0, e_1, e_2, \cdots, e_n\}$ is a maximal multiplicatively cosed set in M.*

Proof. Suppose, on the contrary, that there is a multiplicativery colsed subset G of M which properly contains $\{0, e_1, e_2, \cdots, e_n\}$ and let $f \in G \setminus \{0, e_1, e_2, \cdots, e_n\}$. Since e_1fe_2 is a nilpotent element, e_1fe_2 must be 0. Similarly we have $e_1fe_i = 0$ for $i = 3, \ldots, n$. Hence we have $e_1f(1-e_1) = e_1fe_2 + \cdots + e_1fe_n = 0$. Similarly we have $(1 - e_1)fe_1 = 0$. Therefore $e_1f = e_1fe_1 = fe_1$, that is e_1 and f are commutative.

By the same way, we can see that f and e_i are commutative for $i = 2, \cdots, n$. Now we can easily see that $e_1 f, e_2 f, \cdots, e_n f$ are primitive orthogonal idempotents. Since $1 = e_1 f + \cdots +_n f$ and since f is primitive, we conclude that $f = e_i f$ for i. Since f and e_1 are commutative, $e_1 f$ and $e_1(1-f)$ are orthogonal idempotents. Since $e_1 = e_1 f + e_1(1-f)$ and since f is primitive, we see $e_1(1-f) = 0$. Then $e_1 = e_1 f = f$, a contradiction.

Example 3. *Consider the ring $R = \mathbf{Z} + M_2(\mathbf{Q}[x]x)$. R is an order of $M_2(\mathbf{Q}[x])$. We can easily see that the idempotents of R are only 0 and 1.*

Theorem 2. *Let R be a ring and let M denote the set consisting of all primitive idempotents in R and zero. Suppose that 1 is a sum of primitive orthogonal idempotents. Then M is closed under multiplication if and only if R is a direct sum of rings with no non-trivial idempotents.*

Proof. Suppose that M is closed under multiplication and that there are primitive orthogonal idempotents $e_1, e_2, \cdots, e_n$ of R such that $1 = e_1 + e_2 + \cdots + e_n$. Since $\{0, e_1, e_2, \cdots, e_n\}$ is a maximal multiplicatively cosed set in M by Theorem 1, we conclude that $M = \{0, e_1, e_2, \cdots, e_n\}$. Then $e_1, e_2, \cdots, e_n$ are central orthogonal idempotents and $R = e_1 R \oplus \cdots \oplus e_n R$. Since each e_i is primitive, each $e_i R$ has no non-trivial idempotents.

In [2], D. Dolžan proved that M is closed under multiplication if and only if R is a direct sum of local rings([2, Corollary 5.6]). Now we generalize this result to semiperfect rings. Let R denote a ring and J denote its Jacobson radical. A ring R is called semiperfect if R is semilocal and idempotents of R/J can be lifted to R. All basic results concerning rings can be found in [1].

If R be a semiperfect ring, then there are primitive orthogonal idempotents $e_1, e_2, \cdots, e_n$ of R such that $1 = e_1 + e_2 + \cdots + e_n$ and each $e_i R e_i$ is a local ring. Hence we have the following.

Corollary 1. *Let R be a semiperfect ring and M be the set of all minimal idempotents and zero in R. Then the set M is closed under multiplication if and only if R is a direct sum of local rings.*

Let $[M]$ denote the set $\{eR \mid e \in M\}$, that is, $[M]$ is the set of right ideals of the form eR for some primitive idempotent e and the ideal 0.

Theorem 3. *Let R be a semiperfect ring and $[M]$ be the set of right ideals of the form eR for some primitive idempotent e and the ideal 0. Then the set $[M]$ is closed under multiplication if and only if R is a finite direct sum of matrix rings over some local ring.*

Proof. If R is a finite direct sum of matrix ring over some local ring, then clearly M is closed under multiplication. Let e and f be two primitive idempotents of R. Then either $eRfR = 0$ or $eRfR = gR$ for some primitive idempotent $g \in R$. If $eRfR = 0$, then $(fReR)^2 = 0$. In this case $fReR$ is not a nonzero direct summand of R, and so we conclude that $fReR = 0$. If $eRfR = gR$ for some primitive idempotent $g \in R$, then $eR \supseteq gR$. Using modular law, we have $eR = eR \cap (gR \oplus (1-g)R) = gR \oplus eR \cap (1-g)R$. Since eR is indecomposable, we conclude that $gR = eR$. Thus $eRfR = eR$, and so $eRfRe = eRe$. Then we can write $e = \sum_{i=1}^{n} ea_i fb_i e$ for some $a_i,\ b_i \in R$. Since eRe is a local ring, for some k, $ea_k fb_k e$ is invertible in eRe. Similarly there exists $c,\ d \in R$ such that $fcedf$ is invertible in fRf. These mean that $eR \cong fR$. Since R is semiperfect, $R = e_1R \oplus \cdots \oplus e_nR$ for some primitive idempotents $e_1, \cdots, e_n$. By the fact proved above, $R = R_1 \oplus \cdots \oplus R_m$ such that each two-sided ideal R_i is a finite direct sum of isomorphic indecomposable modeles. Then $R \cong End(R_1) \oplus \cdots \oplus End(R_m)$. Thus each $R_i \cong End(R_i)$ is a matrix ring ove a local ring.

References

[1] F. W. Anderson and K. R. Fuller: Rings and Categories of Modules, Second Edition, Springer-Verlag, New York-Heidelberg-Berlin, 1992.

[2] Dolžan, D.: Multiplicative sets of idempotents in a finite ring, J. Algebra 304 (2006), no. 1, 271–277.

A Toronto Space

Philip Stephen Doi

January 4, 2019

Abstract

A Toronto space is a topological space homeomorphic to each of its subspaces of the same cardinality. What follows is a brief introduction to the Toronto problem, which asks if there is a nondiscrete Hausdorff Toronto space of cardinality $\aleph_1$. Details of more general Toronto spaces will be eschewed in favor of covering the basic consequence of postulating the existence of a nondiscrete Hausdorff Toronto space.

1 Introduction

A topological space is Toronto if it is homeomorphic to each of its full-cardinality subspaces. Though consistent examples exist of Toronto spaces and of their generalizations [1], it remains an open problem to determine if there is a non-discrete Hausdorff Toronto space of cardinality $\aleph_1$. The most up-to-date literature on Toronto spaces and the related problem can be found in [2]. As shown in [3, 4], problems about these spaces, along with the appropriate generalizations, are usually questions of consistency relative to Zermelo-Fraenkel-Choice (ZFC) set theory and extensions thereof.

The intention of this paper is to give a quick overview of the Toronto problem and the basic results surrounding it. The main section is Section 2 with Section 3 being concerned with concluding remarks and acknowledgments. The coverage of topics will closely follow the presentation on Toronto spaces in The Fourth Annual Conference for the Exchange of Mathematical Ideas. The important points and results found in the lore will be central in this overview, although a few basic results proven in the course of undergraduate research will be listed; these are likely to be obvious to the specialist. With that being said, this paper is intended for the more casually interested, hopefully affording a quick reference. Since research topics in point-set topology do not seem very prominent, is always good to be given the opportunity to promote an interesting problem.

10.5281/zenodo.2532065

2 Previous Results with Discussions

For the rest of the paper, assume that X is a nondiscrete Hausdorff Toronto space. We will define the n^{th} derivative on X by transfinite recursion

$$X^{(0)} = X,$$

$$X^{(n+1)} = X^{(n)} - I_n^X \text{ where } I_n^X = \left\{ x \in X^{(n)} \mid x \text{ is isolated} \right\},$$

and if δ is a limit ordinal,

$$X^{(\delta)} = \bigcap_{\alpha \in \delta} X^{(\alpha)}.$$

This transfinite sequence of derivatives is a decreasing sequence of closed sets. We then have the following result.

Proposition 1. *The following holds:*

1. *The set I_0^X is infinite.*

2. *The set I_0^X is countable and dense in X.*

3. *For all $\alpha \in \omega_1$, $X^{(n)} \cong X$.*

4. *X is scattered, with derived length ω_1; that is*

$$X^{(\omega_1)} = \emptyset.$$

Since author was given the opportunity to prove these results as part of an undergraduate research project, the proofs will be omitted for sake of the adventurous reader. However, a few comment are in order, since this presentation deviates from the more comprehensive and technical outlines in [2]. Statement (1) requires only the assumption that X is infinite, Hausdorff, and Toronto. Since X is nondiscrete, statement (2) is almost immediate. One can find proofs of (1) and (2) in [5], although note that the author of that paper assumes the Continuum Hypothesis for the proof of (2), which is probably an accident since it is not needed; it is already the case that every infinite subset of $\aleph_1$ is countable or of cardinality $\aleph_1$. Statements (3) and (4) require transfinite induction. It is easy to see that $X^{(n)}$ homeomorphic to X for the finite ordinals as illustrated in Figure 1. It can be shown by homeomorphism that every $I_\alpha^X \neq \emptyset$ when $\alpha \in \omega_1$. Moreover,

$$S = \bigcup_{\alpha \in \omega_1} I_\alpha^X$$

is a scattered space of rank ω_1, necessarily homeomorphic to X.

From Proposition 1.(2) specifically, there is a notable result in the lore on Toronto spaces:

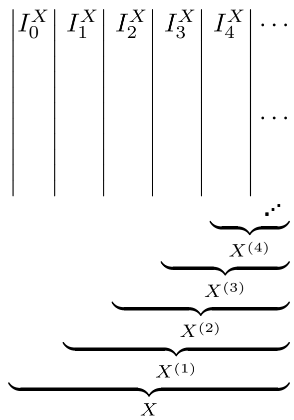


Figure 1: The first few derivatives of X. Each section is the isolated points of some n^{th} derivative.

Proposition 2. *If X exists, then*

$$2^{\aleph_1} = 2^{\aleph_0}.$$

The the proof of this proposition can easily be sketched. For any $S \in \left[X^{(1)}\right]^{\aleph_1}$, there is a homeomorphism

$$f_S : X \longrightarrow S \cup I_0^X.$$

Since the spaces involved are Hausdorff, f_S is uniquely determined by the permutation $\alpha = f\mid_{I_0^X}$ in Aut $\left(I_0^X\right)$, which has cardinality $2^{\aleph_0}$. However, there are $2^{\aleph_1}$many subsets S of $X^{(1)}$ with cardinality $\aleph_1$. One can easily find an injective map from the set of all such homomorphism f into Aut $\left(I_0^X\right)$. For the nonspecialist, note that this result does not pose a problem for the existence of X, since Martin's Axiom is relatively consistent with ZFC set theory and implies this result when $\aleph_1 < 2^{\aleph_0}$. Roughly speaking, Martin's Axiom simply says that cardinals less than the continuum behave like the countably infinite $\aleph_0$.

In discussing the proof of Proposition 2, some peers have suggested a closer look at homeomorphism groups and overall groupoid structure of $[X]^{\aleph_1}$. Moreover, it has been noted that the set of permutations corresponding to f_S, when S is a proper subspace, form a pure subsemigroup of Aut $\left(I_0^X\right)$. In essence, we consider the set of all proper embeddings

$$f : X \longrightarrow X$$

where $I_0^X \subset f(X)$. This forms a pure semigroup and can be mapped into Aut $\left(I_0^X\right) \cong S_{\mathbb{N}}$ by the restriction map.

During the undergraduate research project, a few basic results were proven that could not be found in the available literature.

Proposition 3. *The following holds for X:*

1. *X is Baire.*
2. *X is not metacompact.*
3. *X is not σ-compact.*

Proof. Every family of dense open sets must contain I_0^X, which is dense in X. Therefore, (1) is immediate.

For (2), if X is metacompact, then X must be Lindelöf ; but $\{X - X^\alpha\}_{\alpha \in \omega_1}$ is an open cover of X without a countable subcover.

Kunen's Theorem in [2] demonstrates that $\omega + 1$ cannot be embedded into X. Consequently, X is anti-compact, meaning X is strictly the union of $\aleph_1$ many compact sets. Therefore, X is not σ-compact. □

It is an interesting challenge for an undergraduate to prove the majority of these results without making use of the insights of Dr. Brian's paper.

3 Final Remarks

These are a few things to note that might be of interest for someone looking into related topics in point-set topology:

1. The space X is an example of a D''-Baire space. Such spaces are discussed in [6].
2. It has been shown that if X is regular, then X is a not countably compact. It is to be wondered at if X is countably paracompact. If X is not, then a normal non-discrete Toronto space ($\mathrm{T}_1 + \mathrm{T}_4$ space) would be an example of an $\aleph_1$ Dowker space.
3. The space X is an example of an ω_1-Toronto space; X is homeomorphic to every subspace of the same rank and cardinality. Consistent examples of these spaces exists, but there are none known to have countable levels (I_α^X is countable for all $\alpha \in \omega_1$).
4. So far, it remains unknown if there could regular or normal nondiscrete Toronto spaces; Kunen's result shows that there are no nondiscrete metrizable Toronto spaces in ZFC.
5. The existsence of X is related to resolving a problem in topological partition theory; see Weiss's article on *Paritioning Topological Spaces* in [7].

So far, there have been no additional status updates to the problem.

Special thanks to the good people at the Annuel Conference for Exchange of Mathematical ideas both for their ideas and for their unique and insightful perspectives including Drs. Mupasiri and Gauthier. Also, a thanks goes out to Dr. Huck, from Northern Arizona University, for patiently mentoring the project and taking time out of his busy schedule. Also, a thank you goes out to Embry-Riddle Aeronautical University, Prescott.

References

[1] Gary Gruenhage and J Tatch Moore. Countable toronto spaces. *arXiv preprint math/9902071*, 1999.

[2] William Rea Brian. The toronto problem. *Topology and its Applications*, 162:76–90, 2014.

[3] Elliott Pearl. Open problems in topology. *Topology and its Applications*, 136(1):37–85, 2004.

[4] Elliott M Pearl. *Open problems in topology II*. Elsevier, 2011.

[5] Manuel Rivera. Toronto spaces. *Mathematics Review*, page 24, 2007.

[6] S Garcia-Ferreira, A Garcia-Maynez, and M Hrusak. Spaces in which every dense subset is baire. *Houston J. Math*, 39(1):247–263, 2013.

[7] *Mathematics of Ramsey Theory (Algorithms and Combinatorics)*. Springer, 2012.

Mathematical Aspects of Kolam, a Dravidian Art Form

R. Kit Kittappa
formerly affiliated with Lincoln University PA
rk_kittappa@yahoo.com

Abstract

Kolam is a non representative ancient art form that is still practiced today in South India which is the land of the Dravidians. I will present a definition of a kolam in terms of tiles and rules for using the tiles, kolam's relation with the grid graph and its spanning subgraphs, representation of a kolam with ordered pairs of (0,1)-matrices, boolean algebra of kolams and enumeration of non isomorphic kolams.

1. Introduction

Kolam (*kōlam*) is a class of abstract (non representative) designs drawn by women of South India, the Dravidian land, the land that is recognized as such in the National Anthem of India. It is also the home of unique Dravidian arts like Dravidian temple architecture, Dravidian music system (Carnatic music), Dravidian dance forms (Kuchipudi, Kathakali and Sathir Attam, the last of which is also known as Bharatha Natyam), Dravidian cuisine and Dravidian literature in the languages, Tamil, Malayalam, Kannada, Tulu and Telugu, which are off-shoots of the Proto-Dravidian language known to linguists. Examples of kolam can be seen in in Fig. 1.1.

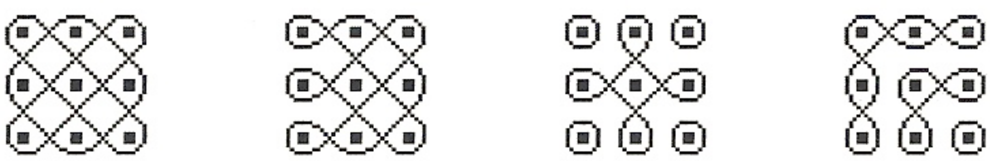


Figure 1.1: Some simple examples of kolam with 3x3 dots

10.5281/zenodo.2532059

The author of this paper has shown, in a paper (he has completed and ready for submission) [4], that kolam is found in the pre-Vedic civilization called the Indus Valley Civilization (3300-1900 BC), that had existed in northeast Afghanistan, Pakistan and northwest India.

In the early mornings, one can find women draw kolam designs on the ground at the entrance to their homes. It can also be found on the floors of halls used for celebration. It is said that originally rice powder was used as a humane gesture to feed ants. Now, white powdered rock is also being used. First dots are placed. Then lines, straight and curved, are drawn going around the dots. This is done by taking a pinch of the powder and letting it go gradually, as the hand is moved.

Kolam is a Tamil word meaning “form” and it has been used at least from the 17th century onwards, as seen from Tamil literary works like *Maturai Mīnāṭci Ammai Kuṟam*, (verse 6) by Kumara Guruparar and *Thiru Kuṟṟāla Kuravañci* (song 56, 1) by Tirikūda Rāsappa Kavirāyar. Professor Rukmani [6] writes that during the period of the ancient Tamil Literary Academy called the Sangam (also the Last Sangam or the Third Sangam), 400 BCE to 200 CE, kolam was called *vari* (Tamil: *vari* = line straight or curved). Use of this term can be seen in *Pattiṉapālai* (verse 165) and *Naṟṟiṉai* (verses 123 and 378).

2. Definition of kolam tiles and kolam

We will set down definitions for k-tile and kolam. (k-tile is a temporary name to be revised soon.)

<u>Definition 2.1:</u> S is a set of k-tiles iff

(a) It is a set of non overlapping squares of the same size.

(b) The center of each square is marked by a (conspicuous) **dot**.

(c) None, one or more than one of the midpoints of the sides of the squares may be marked. (These marked midpoints will be called **attracting points**).

(d) If a square has no attracting points then a circle is drawn within the square, with its center at the center of the square. (It is considered that the circumference is an elastic band that stays tightly around the circle considered to be a cylinder). If there are attracting points then the circumference of the circle (the elastic band) is stretched around the attracting points.

The above definition and the number of ways of choosing attracting points on the sides of a square generate the following 6 kinds of k-tiles (Figure 2.1).

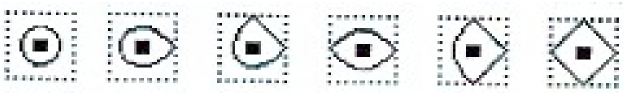


Figure 2.1: Six kinds of k-tiles.

Definition 2.2: Two k-tiles are **connected** iff

(i) both have a common side and

(ii) the common side has either no attraction point or exactly one attraction point.

Rules for putting the k-tiles together

(a) each k-tile should be connected to one or more tiles such that the dots of all the tiles form a rectangular array,

(b) every attraction point in a k-tile should fall on the attraction point on another k tile,

(c) k-tiles can be rotated through multiples of 90° and

(d) k-tiles of each kind should be available as needed.

Rules (a) and (b) imply that there will be a rectangular border for the k-design without any attracting points on it.

Definition 2.3: A configuration is a kolam iff it is constructed with the k-tiles in Definition 2.1 using Rules (a)-(d) above.

We will now revise the name k-tile **kolam tile**. (We did not want to call a tile as kolam tile before defining a kolam).

If a kolam has an array of mxn dots its **size** will be mxn.

The configuration within a tile will be called an **atom** of the kolam. The atom in the tile with no attraction point (Figure 2.1, the left most tile) will be called **circle-dot** for convenience.

Definition 2.4: Two atoms are connected iff their tiles are connected or equivalently if the atoms share an attraction point.

Figure 2.2 shows examples of kolams with tiles.

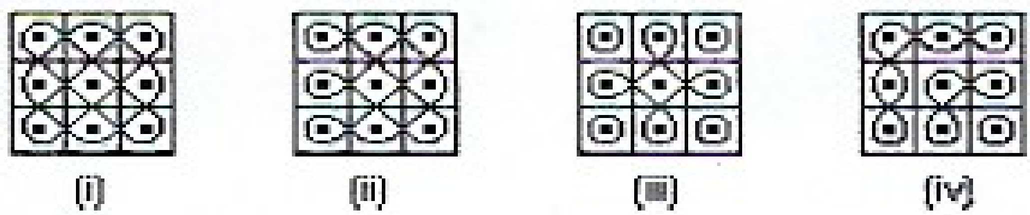
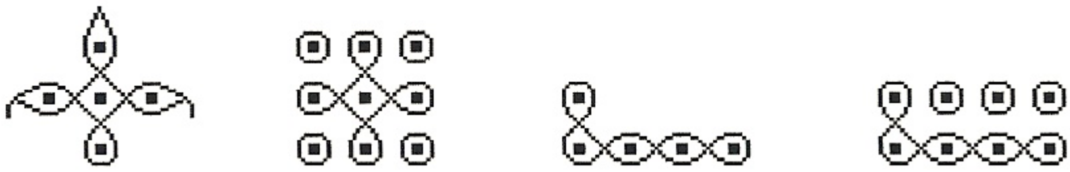


Figure 2.2: Examples of kolams with tiles. (The same kolams in Figure 1 are used.)

The kolam in Figure 2.2 (i) has the maximum possible number of attracting points and the others have lesser number of attracting points. This kolam is a unique 3x3 kolam that will be called the 3x3 **full kolam**, FK_{3x3} and in general FK_{mxn}. The kolam with no attracting points but only with dot-circles (not shown above) will be called a **null kolam** NK_{mxn}.

If the rotation in Rule (b) is not allowed there will be 16 kinds of tiles instead of 6. Many, especially computer scientists, have used kolam tiles without explicitly defining the kolam tile or a kolam. It is difficult to trace who mentioned the kolam tiles first. However we will refer the reader to a paper by Kawai et al [3].

Sometimes, the kolam artists use an empty tile in the place of the tile with the dot-circle. Also, they may add embellishments. However, for a mathematical discussion we need to replace an empty tile with the tile that has a circle and a dot.

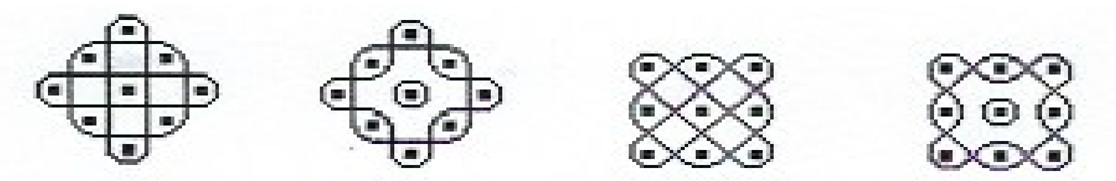


and trim off any embellishments as shown in Figure 2.3, where tiles are not shown because artists do not show tiles. We will say the resulting kolam is **standard form**

(i) (ii) (iii) (iv)

Figure 2.3: Changing artist's kolams to standard forms [(i) to (ii) and (iii) to (iv)].

Figure 2.4: A 45° tilt coverts (i) to standard form (iii) and (ii) to standard form (iv).



About the middle of the last century, kolam competitions began to appear. Unfortunately, in most cases, the judges were celebrities like movie stars and politicians, who were not familiar with kolam as a special art. They gave prizes to the few drawings that represented flowers, animals, Hindu religious representations or geometrical drawings. So, a part of the population began to believe that any picture drawn on the ground is a kolam. See Figure 2.5 for some pictures drawn on the ground and believed wrongly by some to be kolams.

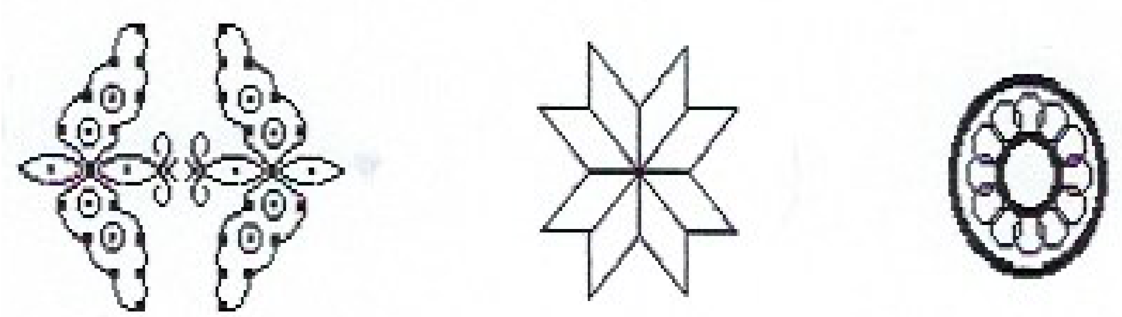


Figure 2.5: Examples of pictures drawn on the ground that are not kolams. The right most one is done with colored paste and is a North Indian art form called Rangoli.

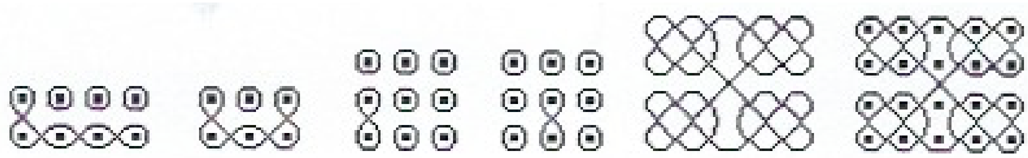
People began to call kolam as *kambi* kolam (Tamil: *kambi* = wire) and *pulli* kolam (Tamil: *pulli* = dot) just to distinguish the real kolam from mere pictures drawn on the ground. All kolam artists, who were contacted, agreed that the real kolam are the pulli kolam and that other pictures drawn on the ground are not kolams.

3. Graph theoretical considerations

A kolam, without the dots, is a 4-regular pseudograph (a graph having loops and/or multiple edges) with the attracting points as its vertices. However, even after the dot is removed from the 1x1 kolam (which will be just the dot-circle), it is not a pseudograph unless we introduce a vertex on the circle. Other differences are described in the legend for Figure 3.1.

Definition 3.1: Let S be a set of atoms of a kolam K.

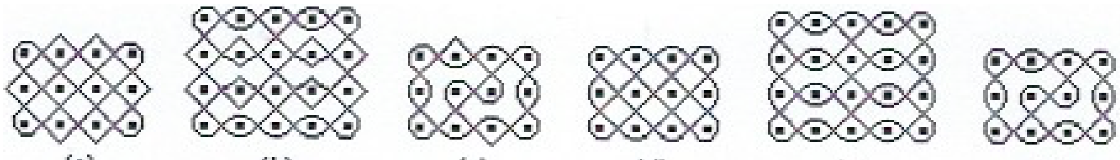
S is a **component** of K iff it is a maximally connected set.



(i) (ii) (iii) (iv) (v) (vi)

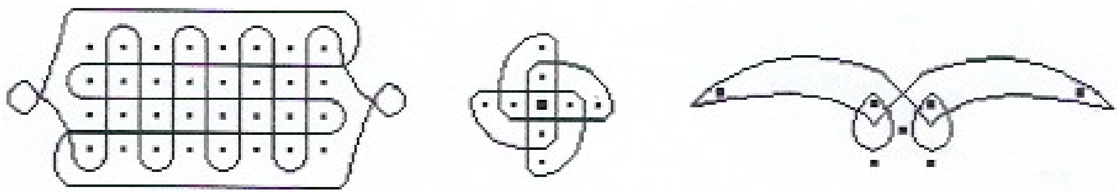
Figure 3.1: The 5 atom components in the kolams in (i) and (ii) are equivalent as pseudographs, when the dots are removed, but they are not equivalent as kolam components. In fact, they are components in kolams of different sizes, namely 2x4 and 2x3. The two atom components in (iii) and (iv) are equivalent as pseudographs but not as components of the two different 3x3 kolams because they occupy different positions. The figure in (v) is a Celtic knot. Even after introducing dots as in (vi), it is not a kolam because the middle part cannot be represented by kolam tiles. There are a few Celtic knots that will be a kolam when dots are introduced.

Some of the sand drawings of the "sona" tradition of the Chokwe people of Angola (see Gerdes [2]) can be changed to kolams (Figure 3.2), after making minor changes. They are drawn by men to tell stories whereas kolams are drawn by woman for decoration.



(a) (b) (c) (d) (e) (f)

Figure 3.2: (a), (b) and (c) are sand drawings of the Chokwe people that are not kolams but with some changes they have been transformed into kolams (d), (e), (f) respectively. (a) is a plaited mat design, (b) is lion's stomach and (c) is chased chicken.



(i) (ii) (iii)

Figure 3.3: Chokwe sand drawings that cannot be changed to kolams. (i) A representation of a game board (ii) a spider (iii) a bat

Each component of a kolam, without its dots, is an Eulerian pseudograph because of its 4-regularity.

We will show that each kolam corresponds to a unique spanning subgraph of a grid graph. For this purpose, we will now recall what are grid graphs and spanning subgraphs.

Definition 3.2: Let m and n be positive integers greater than 1. G_{mxn} is a **grid graph** iff G_{mxn} is the cross product of two paths of lengths (m-1) and (n-1).

For convenience, it is often represented by an (m-1)x(n-1) rectangular array of contiguous squares. The mxn vertices of the squares are the vertices of G_{mxn}. The edges are the (m-1)n+m(n-1) or 2mn-m-n line segments joining the vertices (see Figure 3.2 (i) with m=n=3).

Definition 3.3: H is a **spanning subgraph** of a graph G iff (a) V(H) = V(G) and (b) E(H) ⊆ E(G), with V standing for vertex set and E for edge set.

Note that this definition implies that a graph G is itself a spanning subgraph.

Figure 3.2: (i) G_{3x3} (i)-(iv) A few spanning subgraphs of G_{3x3}. Solid lines are edges.

Suppose that, in an mxn kolam, we draw 0, 1 or more line segments joining each pair of dots in two connected atoms. (The line segment goes through the attraction point in between.) We get a spanning subgraph of G_{mxn}. If you compare Figure 1 and Figure 3.2, the spanning subgraphs (i)-(iv) in Figure 3.2 are the spanning subgraphs obtained from the kolams (i)-(iv) in Figure 1 respectively by joining the dots, as described above. It may appear that to count the number of kolams of size mxn, we only need to count the number of spanning subgraphs of G_{mxn}, which is in fact the size of the power set of $E(G_{mxn})$. However, that will count isomorphic kolams more than once. We will take up the enumeration of kolams in Section 5.

In Figure 2.2, the kolams in Figure 1 are shown with tiles. If we carry out the joining of dots described above on kolams in Figure 2,2, we get the grid graph with tiles (Figure 3.3).

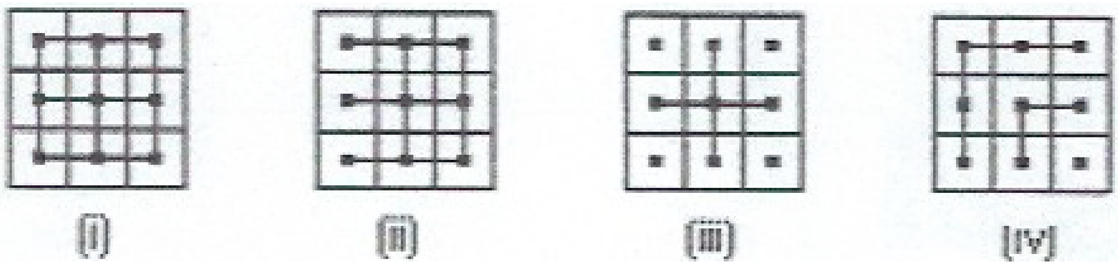


Figure 3.3: Spanning subgraphs corresponding to kolams in Figure 3.2, with tiles.

This gives us the idea that it is possible to have tiles that can generate spanning subgraphs of G_{mxn}. Figure 3.4 (second row) shows the possible such tiles.

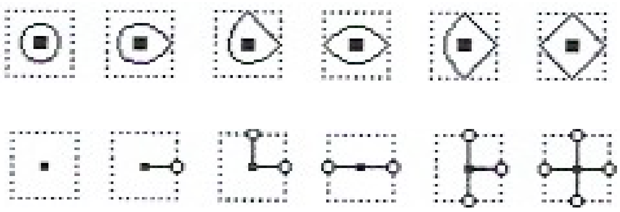
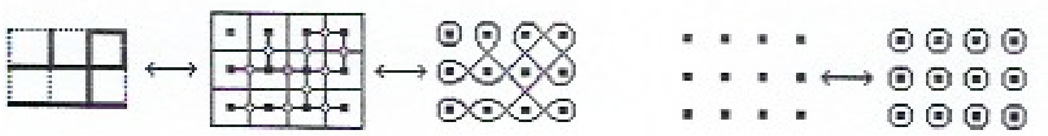


Figure 3.4: Kolam tiles are in the first row and tiles for the spanning subgraphs of a grid graph are in the second row. Corresponding tiles are placed one below the other. Attracting points on the new kind of tiles are shown, for comparison,

Given a spanning subgraph of G_{mxn}, we can construct a unique kolam and vice versa (Figure 3.5).



(i) (ii)

Figure 3.5: (i) A spanning subgraph of G_{3x4} transforms into a kolam. (ii) The null spanning subgraph of G_{3x4} transforms into the null kolam, NK_{3x4}.

This shows there is a one-to-one correspondence between the set of all kolams of size mxn and the set of all spanning subgraphs of G_{mxn}. We can, in fact, define mxn kolams in terms of spanning subgraphs of G_{mxn}.

Also, a kolam can be constructed with tiles that has just the attracting points (Figure 3.6 and 3.8 (i)). In fact, the attracting points are the essential entities in all the different kinds of tiles.

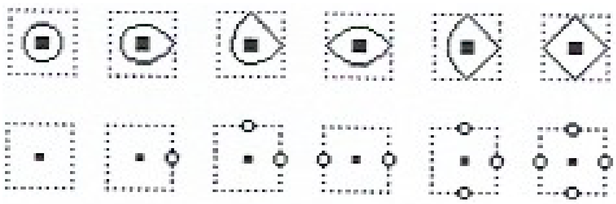


Figure 3.6: Kolam tiles are in the top row. Tiles with attracting points only are placed below the corresponding kolam tiles.

It is possible to define somewhat different kolam tiles that have the same atoms but have repulsing sides instead of attracting points. For each tile in Figure 2.1, highlight each side that does not have an attraction point and consider that these sides repulse the circumference to maintain its circular shape. If all four sides are repulsing, they maintain the atom as a dot-circle. If a side is not repulsing, then the circumference is attracted towards its midpoint. The same atoms are formed but with repulsing sides instead of attracting points (Figure 3.7). When using these tiles, the outermost rectangular border of the contiguous rectangular array of squares must be considered to be consisting of repulsing sides. (See Figure 3.8)

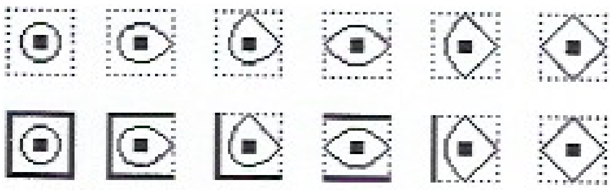
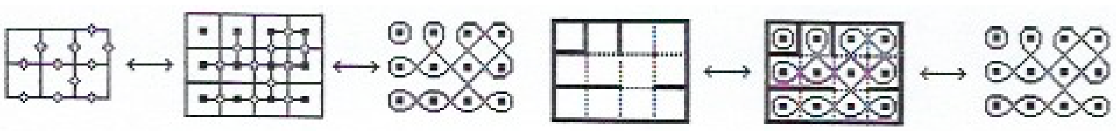


Figure 3.7: The kolam tiles are in the first row and the tiles with repulsing sides are below each corresponding kolam tile.

Suppose, we have an mxn rectangular array of contiguous squares and B be its rectangular border. Also, suppose we arbitrarily choose some of the sides in the interior of B. Then we can construct a unique kolam using the tiles with repulsing sides (Figure 3.8). (Recall it is to be assumed that B consists of repulsing sides.) Hence, there is a 1-1 correspondence between possible choices of sides in the interior of B and mxn kolams.

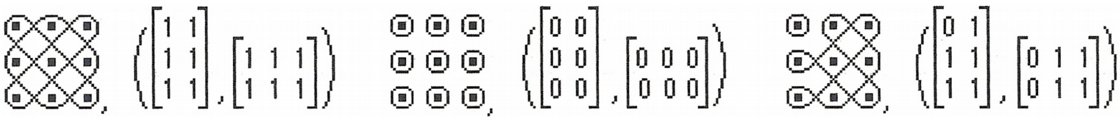


(i) (ii)

Figure 3.8: (i) Constructing a spanning subgraph of G_{3x4} and a 3x4 kolam from a choice of midpoints of the sides of a rectangular array of contiguous squares.
(ii) Using tiles with repulsing sides to construct a 3x4 kolam from a choice of interior sides of a rectangular array of contiguous squares.

4. Matrix representation of a kolam and the algebra of kolams

Consider the attracting points that connect atoms horizontally in the mxn full kolam FK_{mxn}. On the first row there will be n-1 attracting points. There are m such rows. Construct a (0,1)-matrix $H_{mx(n-1)}$ with 1s in the relative position of the attracting points. For kolams that are not full kolams, some attracting points will be missing. Put 0s in the corresponding places. Similarly, construct the (0,1)-matrix $V_{(m-1)xn}$ for the attracting points that connect atoms vertically. The unique matrix ordered pair of (0,1)-matrices, $(H_{mx(n-1)}, V_{(m-1)xn})$ can be constructed for an mxn kolam. and the kolam can be recovered uniquely. For examples see Figure 4.1.



(i) $\left(\begin{bmatrix}1&1\\1&1\\1&1\end{bmatrix},\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\right)$

(ii) $\left(\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\right)$

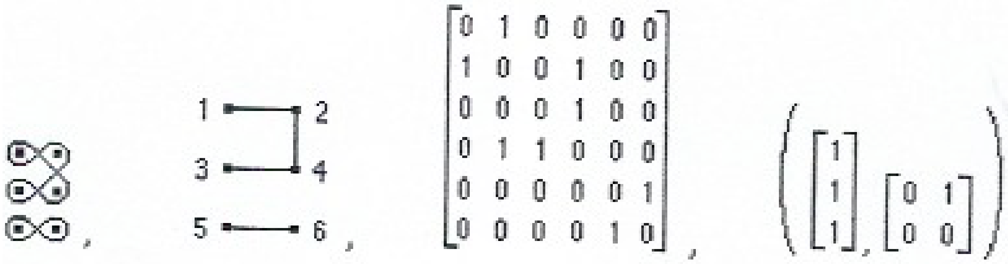
(iii) $\left(\begin{bmatrix}0&1\\1&1\\1&1\end{bmatrix},\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}\right)$

Figure 4.1: Some kolams and their corresponding (0,1)-matrix pairs.

It may appear that the adjacency matrix of the associated spanning subgraph the grid graph may be a better choice than a pair of matrices to represent a kolam. We will show that it is not so. For an mxn kolam the number of vertices in FK_{mxn} is $m(n-1) + (m-1)n = 2mn-m-n$, which will be the same as the total number of entries in the corresponding matrix pair. The number of entries in the adjacency matrix of the associated spanning subgraph of G_{mxn} is $(mn)^2$.

Now, for m,n greater than 1, $(mn-1)^2 \geq 1 \Rightarrow (mn)^2 \geq 2mn > 2mn-m-n$.

See also Figure 4.2



(i) (ii) (iii) (iv)

Figure 4.2: (i) a 3x2 kolam, (ii) the spanning subgraph of G_{mxn} corresponding to (i) with its vertices having number labels, (iii) the adjacency matrix of graph in (ii), (iv) the matrix pair for kolam in (i)

Let B_2 be the boolean algebra $(B_2, +, \bullet, 0,1)$ with $B_2 = \{0,1\}$. Let A, B, C, D be mxn (0.1)-matrices.

Let $A \cup B = C \Leftrightarrow c_{ij} = a_{ij} + b_{ij}$, $A \cap B = C \Leftrightarrow c_{ij} = a_{ij} \bullet b_{ij}$, $A' = C \Leftrightarrow c_{ij} = a_{ij}'$,

U = the mxn (0,1)-matrix with all 1s and Φ = the mxn (0,1)-matrix with all 0s.

Then it can be verified that the set M of mxn (0,1)-matrices will form the boolean algebra $(M, \cup, \cap, ', U, \Phi)$.

Now consider the set P of all ordered (0,1)-matrix pairs with all first entries of the same size and all second entries of the same size. For elements in P, with A and C of the same size and B and D of the same size, we define

$(A, B) \cup (C, D) = (A \cup B, C \cup D)$, $(A, B) \cap (C, D) = (A \cap B, C \cap D)$, $(A, B)' = (A', B')$, where the operations on the right side of the equations are as defined earlier.

Let us also define the matrix pair U and Φ in P as U = the (0,1)-matrix pair with all 1s and Φ = the mxn (0,1)-matrix pair with all 0s. Then we can verify that set P forms the boolean algebra, $(P, \cup, \cap, ', U, \Phi)$, where the operations are as in the right side of the equations above.

This provides operations for the algebra of kolams and related configurations.

With these operations and the one-to-oneness of between set of mxn kolams and the associated configurations, it is possible to set up isomorphisms between them.

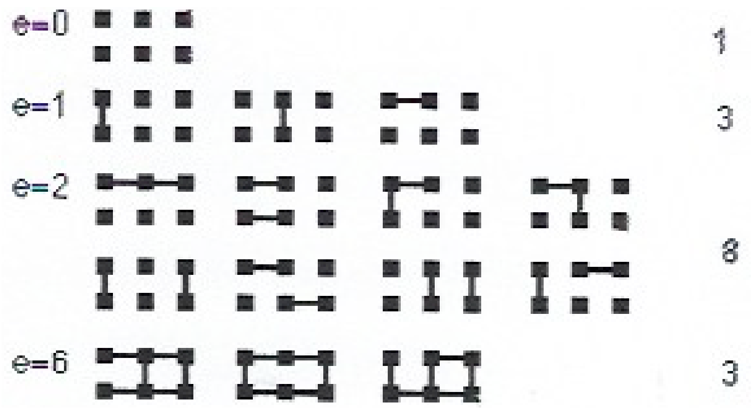
5. Enumeration of kolams

To carry out enumeration of non isomorphic mxn kolams, we can instead enumerate the non isomorphic spanning subgraphs of G_{mxn}. because of the 1-1 property between them. The Polya's method [1] that results in a generating function will be suitable. Usually, the permutation corresponding to symmetries are considered and they are factored into cycles. This enables one to write down the cycle index. That leads to finding generating functions. Here are few results.

Case m=2, n=3, (m≠n, non square rectangle)

$f(2, 3, z) = 1 + 3z + 8z^2 + 12z^3 + 12z^4 + 8z^5 + 3z^6 + z^7$, and $T(2, 3) = 48$

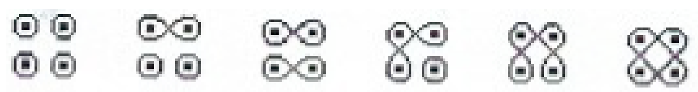
T stands for the total number of non isomorphic configurations. The power of z indicates the number of vertices of the kolam or the number of edges (e) of the spanning subgraph of G_{2x3}. The coefficients indicate the number of non isomorphic configurations. Shown below are non isomorphic spanning subgraphs of G_{2x3} for some values of e.



For m=n, the dots form a square. The generating function for n=2 is

$g(2, z) = 1 + z + 2z^2 + z^3 + z^4$, and $T_S(2) = 6$, where $T_S(2)$ stands for the total non isomorphic (square) kolams of size 2x2.)

We verify the coefficients with kolams. Number of vertices are indicated by v.



v = 0 1 2 2 3 4

The problem of counting the number of non isomorphic spanning subgraphs of G_{mxn} can be considered as one of counting the reversible arrangements of beads of two colors (white and black). The counting of two color one dimensional such arrangements of beads (1xn case) has been solved by Losanitch [5] in his research of chemical structures.

7. Kolam and its relations to knots and links

Though I had presented some preliminary results in this area at the conference, presentation is not being considered till more complete results are obtained.

References

1. Carnegie Mellon University, Polya'sTheory of Counting, https://www.math.cmu.edu/~af1p/Teaching/Combinatorics/Slides/Polya.pdf
2. Gerdes, Paulus, 2007, *Drawings from Angola*, Lulu.com.
3. Kawai, Yasuo, Kosuke Takahashi and Shojiro Nagata, 2007, PsyKolo3D --- Interactive Computer Graphical Content of “kolam” Design, Forma, 22, 113-118
4. Kittappa, Rethinasamy, 2018, Kolam a Dravidian Art form in Indus Valley Civilization, paper completed and is to be submitted.
5. Losanitch, S. M., 1897 Die Isometrie-Arten bei den Homologen der Paraffin-Reihe, Chem. Berlin, 30, 1917-26.
6. Professor Rukmani, 2016, Tamiḻar paṇpāṭṭil kōlaṅkaḷ, Siragu. http://siragu.com/author/rukmani/

Introduction to Graph Pebbling and Rubbling

Nándor Sieben

Northern Arizona University

Abstract

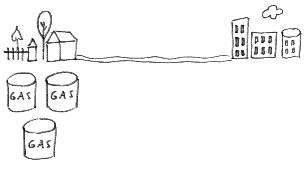
A graph pebbling move removes two pebbles from a vertex of a graph and adds one pebble to an adjacent vertex. In graph rubbling an additional move is allowed that adds a pebble at a vertex after the removal of one pebble each at two adjacent vertices. A vertex is reachable if a pebble can be moved to the vertex using pebbling/rubbling moves. We study the reachability of vertices under different requirements. After a short introduction to pebbling, the known results about rubbling are summarized.

1 Graph pebbling

Graph pebbling was originally used as a method for solving a number theory conjecture of Erdős and Lemke. Starting points to the now large literature can be found in [9, 10] and in the more recent [11]. The friendly web page [8] contains many definitions, results, and a large collection of printable articles. A database of pebbling numbers is available at [18].

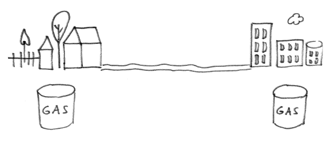
1.1 Introduction

Graph pebbling is a simple model for the transfer of consumable resources. Let's consider a country where old trucks are plentiful but fuel is scarce. Fuel is stored in unit barrels and transporting a barrel from one city to a neighboring city consumes one barrel of fuel. The transfer of a barrel is depicted in the following figures:



before transfer

during transfer



after transfer

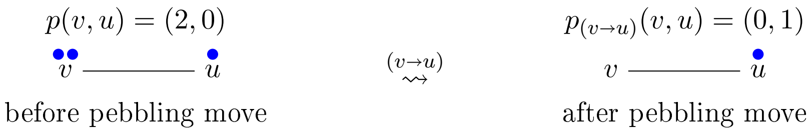
Note that the number of fuel barrels has been decreased during the transportation since the fuel was converted to smoke. We now give a more formal definition.

Definition 1.1. Let u and v be adjacent vertices of a graph with vertex set V. A *pebbling move* $(v \to u)$ removes two pebbles at v and adds a pebble at u.

A pebbling move changes the pebble distribution $p : V \to \mathbb{N}$. The resulting pebble distribution $p_{(v \to u)} : V \to \mathbb{N}$ satisfies $p_{(v \to u)}(v, u, *) = (p(v) - 2, p(u) + 1, p(*))$.

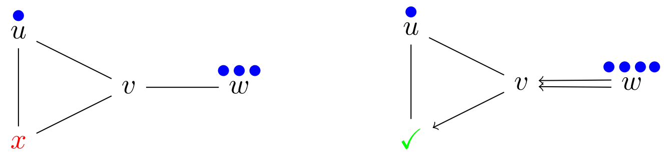
Example 1.2. An example of a pebbling move is shown below:

10.5281/zenodo.2579475



Definition 1.3. The *pebbling number* of a graph G is the minimum number $\pi(G)$ of pebbles in a pebble distribution that makes any vertex reachable, no matter how the pebbles are placed on the vertices.

Example 1.4. The first figure below shows a pebble distribution with 4 pebbles from which vertex x is not reachable. The second figure shows that an additional pebble on vertex w makes vertex x reachable. It is easy to see that every vertex is reachable from any pebble distribution with 5 pebbles, and so the pebbling number of the graph is $\pi(G) = 5$.



We present the pebbling numbers of some basic graph families.

Proposition 1.5.

1. $\pi(P_n) = 2^{n-1}$
2. $\pi(K_n) = n$
3. $\pi(W_n) = n$
4. $\pi(K_{m,n}) = m + n$
5. $\pi(Q_n) = 2^n$
6. $\pi(\text{Petersen}) = 10$

We have some trivial and some more involved bounds on the pebbling number.

Proposition 1.6. *Let G be a graph, d be the diameter of G, n be the number of vertices of G, and $\gamma(G)$ be the domination number of G. Then*

1. $n \leq \pi(G)$
2. $2^d \leq \pi(G)$
3. [3] $\pi(G) \leq (n - d)(2^d - 1) + 1$
4. [3] $\pi(G) \leq (n + 2\gamma(G))2^{d-1} - \gamma(G) + 1$

1.2 The No Cycle Lemma

We formalize the intuition that moving pebbles around in a circle is not helpful for reaching vertices.

Definition 1.7. Let G be a graph with vertex set V. The *transition digraph* of a pebbling sequence (multiset) on G has vertex set V. Every pebbling move $(v\to u)$ adds an arrow $v \longrightarrow u$ to the transition digraph. A pebbling sequence (multiset) is *acyclic* if there is no cycle in the transition digraph.

Lemma 1.8 (No Cycle)**.** *The following are equivalent:*

1. *Vertex v is reachable from the pebble distribution p.*
2. *There is a multiset S of moves such that $p_S \geq 1_{\{v\}}$.*
3. *There is an acyclic multiset R such that $p_R \geq 1_{\{v\}}$.*

4. Vertex v is reachable from p through an acyclic rubbling sequence.

Note that $1_{\{v\}} : V \to \{0,1\}$ is the indicator function. Condition $p_S \geq 1_{\{v\}}$ says that p_S is a pebble distribution ($p_S \geq 0$), and S moved a pebble to vertex v ($p_S(v) \geq 1$).

The No Cycle Lemma shows that we do not have to worry about the ordering of the pebbling moves in a pebbling sequence. If the resulting pebble function is actually a pebble distribution, then an appropriate ordering of the pebbling moves exists.

1.3 Tree graphs and cycle graphs

The *length sequence* of a path partition of a tree is the decreasing finite sequence built from the lengths of the paths in the partition. We order path partitions using the lexicographic order on their length sequences.

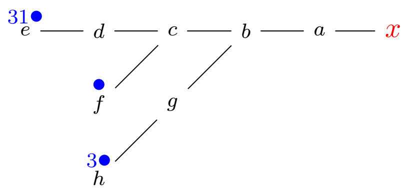
Proposition 1.9 ([4]). *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for a tree T, then*

$$\pi(T) = \sum_{i=1}^{m} 2^{p_i} - m + 1.$$

Example 1.10. The length sequence of the maximum path partition $\{\{x,a,b,c,d,e\},\{b,g,h\},\{c,f\}\}$ of the tree T below is $(5,2,1)$. Hence

$$\pi(T) = (2^5 - 1) + (2^2 - 1) + (2^1 - 1) + 1 = 36.$$

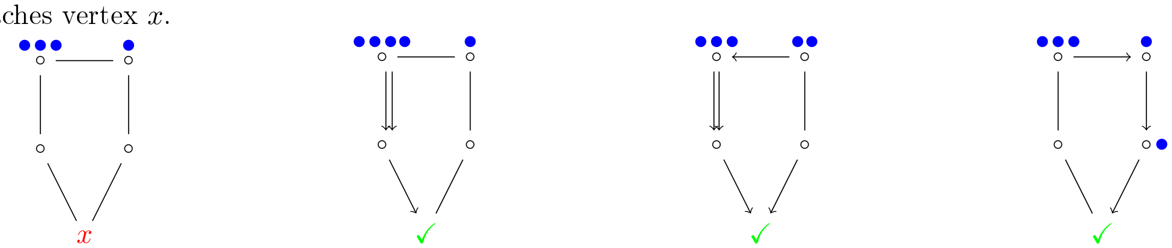
The figure shows a pebble distribution with 35 pebbles from which vertex x is not reachable.



Proposition 1.11 ([15]). *The pebbling number of the cycle graphs are*

$$\pi(C_{2k}) = 2^k \qquad \text{and} \qquad \pi(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1.$$

Example 1.12. The figure below illustrates that the pebbling number of the cycle graph C_5 is $\pi(C_5) = 5$. The arrows represent the arrows in the transition digraph of a pebbling sequence that reaches vertex x.



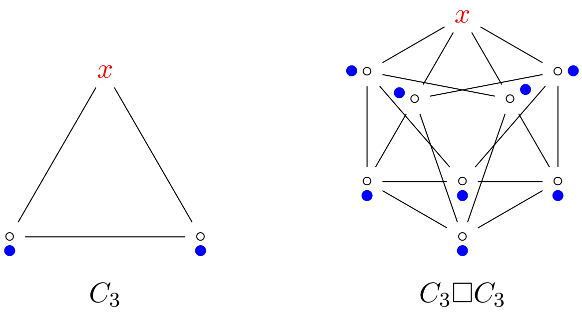
1.4 Graham's Conjecture

The following generated a lot of interest.

Conjecture 1.13 (Graham)**.** *Every product graph $G\Box H$ satisfies $\pi(G\Box H) \leq \pi(G)\pi(H)$.*

Example 1.14. The figure below shows C_3 and $C_3\Box C_3$. Graham's conjecture holds since

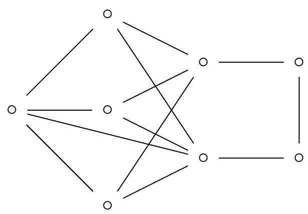
$$\pi(C_3\Box C_3) = 9 \leq 9 = 3 \cdot 3 = \pi(C_3)\pi(C_3).$$



x is not reachable

Graham's Conjecture has been verified for many graph families. An important tool in these investigations is the 2-pebbling property. The following graph does not have the 2-pebbling property.

Example 1.15. The Lemke graph L, shown below, is the smallest feasible counterexample for Graham's Conjecture. Unfortunately $L\Box L$ is a bit large for computer aided pebbling experimentation. Note that $\pi(G) = 8$.



1.5 The t-pebbling number

In a larger emergency we may want to transfer more than one barrels of fuel to a given location.

Definition 1.16. The *t-pebbling number* of a graph G is the minimum number $\pi_t(G)$ of pebbles in a pebble distribution that makes any vertex reachable with t pebbles, no matter how the pebbles are placed on the vertices.

Proposition 1.17 ([4])**.** *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for T, then*

$$\pi_t(T) = t\sum_{i=1}^{m} 2^{p_i} - m + 1.$$

Example 1.18. The t-pebbling number of the wheel graph W_5 is

$$\pi_t(W_5) = \begin{cases} 5, & t = 1 \\ 4t, & t \geq 2. \end{cases}$$

The figures below show the pebble distributions that give the lower bound for $t \in \{1, 2\}$.

$t = 1$ $\qquad$ $t = 2$

Theorem 1.19 ([7]). *There is a t_0 such that the function $t \mapsto \pi_t(G)$ is linear for $t \geq t_0$.*

Question 1.20. *Is it true that $\pi_t(G)$ is the maximum of a few functions that are linear in t, that is, $\pi_t(G) = \max\{a_1 + b_1 t, \ldots, a_k + b_k t\}$?*

1.6 Optimal pebbling

In a well organized country, scientists work hard to store the fuel reserves at the best locations to minimize the size (and the cost) of the reserve.

Definition 1.21. The optimal pebbling number of a graph G is the minimum number of pebbles $\pi_{\text{opt}}(G)$ in a well-chosen pebble distribution from which any vertex is reachable.

Example 1.22. The figure below shows optimal pebble distributions.

$\pi_{\text{opt}}(P_6) = 4$ $\qquad$ $\pi_{\text{opt}}(P_7) = 5$

Now we present the optimal pebbling numbers of some basic graph families.

Proposition 1.23.

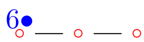
1. [15] $\pi_{opt}(P_n) = \lceil \frac{2n}{3} \rceil$
2. [2] $\pi_{opt}(C_n) = \lceil \frac{2n}{3} \rceil$
3. $\pi_{opt}(K_n) = 2$
4. $\pi_{opt}(W_n) = 2$
5. $\pi_{opt}(K_{m,n}) = 3$
6. $\pi_{opt}(\text{Petersen}) = 4$

1.7 Cover pebbling

Every town might need fuel at the same time in a large emergency.

Definition 1.24. The *cover pebbling number* of a graph G is the minimum number $\pi_{\text{cov}}(G)$ of pebbles to make any vertex reachable at the same time, no matter how the pebbles are distributed on the vertices.

Example 1.25. The cover pebbling number of the P_3 is $\pi_{\text{cov}}(P_3) = 7$. The figure below shows a distribution that proves the lower bound on the cover pebbling number.



Note that all the pebbles are on a single vertex. This in no accident. The following Stacking Theorem is one of the most important results about cover pebbling.

Theorem 1.26 ([20]). *The largest unsolvable pebble configurations contain every pebble on a single vertex.*

1.8 Computational complexity

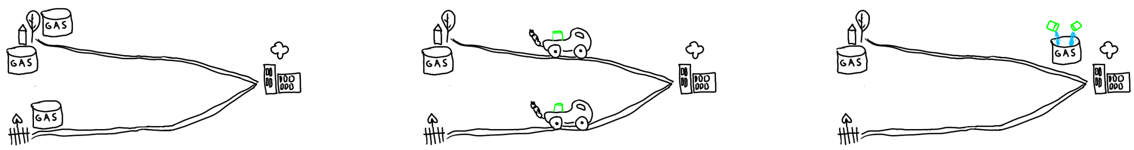
Theorem 1.27 ([14])**.** *Deciding whether a vertex is reachable from a given configuration is NP-complete. Deciding whether $\pi(G) \le k$ is Π_2^P-complete.*

2 Graph rubbling

The theory of graph rubbling is much less developed than the theory of pebbling. Most of the pebbling results are waiting to be transformed into rubbling results. Rubbling results are sometimes easier and other times are harder than their pebbling versions. All the known results about rubbling are in [1, 5, 16, 17, 13, 12]. An informal guide is available at [19].

2.1 Introduction

In graph rubbling an additional transfer option is available. Smaller trucks now can transport half a barrel of fuel while burning only half a barrel. Two of these can be used to fill a full barrel at the destination town. The transfer of a barrel is depicted in the following figures.



before transfer | during transfer | after transfer

We now give a more formal definition.

Definition 2.1. Let v and w be vertices adjacent to vertex u in a graph with vertex set V. A *strict rubbling move* $(v, w \to u)$ removes one pebble each at vertices v and w, and adds a pebble at u. A *rubbling move* is a pebbling move or a strict rubbling move.

A strict rubbling move changes the pebble distribution $p : V \to \mathbb{N}$. The resulting pebble distribution $p_{(v,w\to u)} : V \to \mathbb{N}$ satisfies $p_{(v,w\to u)}(v, w, u, *) = (p(v) - 1, p(w) - 1, p(u) + 1, p(*))$.

Example 2.2. An example of a strict rubbling move $(v, w \to u)$ on the pebble distribution p is shown below:

$$p(v, w, u) = (1, 1, 0) \quad \stackrel{(v,w\to u)}{\rightsquigarrow} \quad p_{(v,w\to u)}(v, w, u) = (0, 0, 1)$$

Definition 2.3. The *rubbling number* of a graph G is the minimum number $\rho(G)$ of pebbles in a pebble distribution that makes any vertex reachable, no matter how the pebbles are placed on the vertices.

Example 2.4. The first figure shows a pebble distribution with 3 pebbles from which vertex x is not reachable. The second figure shows that an additional pebble on vertex w makes vertex x reachable. It is easy to see that every vertex is reachable with any pebble distribution with 4 pebbles and so the rubbling number of the graph is $\rho(G) = 4$.

We present the pebbling numbers of some basic graph families.

Proposition 2.5.

1. $\rho(P_n) = 2^{n-1}$
2. $\rho(K_n) = 2$
3. $\rho(W_n) = 4$
4. $\rho(K_{m,n}) = 4$
5. $\rho(Q_n) = 2^n$
6. $\rho(Petersen) = 5$

Note that unlike in the pebbling case, $\rho(G)$ can be smaller than the number of vertices in the graph.

We have the following bounds on the rubbling number.

Proposition 2.6. *If d is the diameter of G, then*

1. $\rho(G) \leq \pi(G)$
2. $2^d \leq \rho(G)$
3. $\rho(G) \leq (n-d+1)(2^{d-1}-1)+2$

Question 2.7. *What property of G makes $\pi(G) = \rho(G)$?*

Question 2.8. *What can we say about the spectrum of rubbling numbers? In particular, are there any gaps in the spectrum?*

2.2 No Cycle Lemma

The No Cycle Lemma holds for rubbling, but we need to adjust the definition of the transition digraph.

Definition 2.9. Let G be a graph with vertex set V. The *transition digraph* of a rubbling sequence (multiset) on G has vertex set V. Every pebbling move $(v, v \to u)$ adds the arrows $v \Longrightarrow u$. Every strict rubbling move $(v, w \to u)$ adds the arrows $v \longrightarrow u \longleftarrow w$. A rubbling sequence (multiset) is *acyclic* if there is no cycle in the transition digraph.

Lemma 2.10 (No Cycle)**.** *The following are equivalent:*

1. *Vertex v is reachable from the pebble distribution p.*
2. *There is a multiset S of moves such that $p_S \geq 1_{\{v\}}$.*
3. *There is an acyclic multiset R such that $p_R \geq 1_{\{v\}}$.*
4. *Vertex v is reachable from p through an acyclic rubbling sequence.*

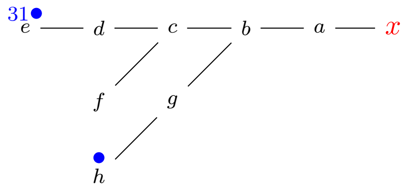
2.3 Tree graphs and cycle graphs

Theorem 2.11. *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for a tree T, then*

$$\rho(T) = 2^{p_1} + \sum_{i=2}^{m} 2^{p_i - 1} - m + 1$$

Example 2.12. The length sequence of the maximum path partition of the tree T below is $(5, 2, 1)$. Hence

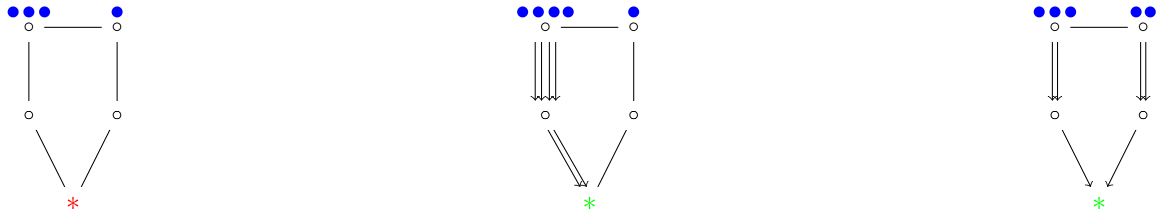
$$\rho(T) = (2^5 - 1) + (2^1 - 1) + (2^0 - 1) + 1 = 33$$



Theorem 2.13. *The rubbling number of the cycle graphs are*

$$\rho(C_{2k}) = 2^k \qquad \text{and} \qquad \rho(C_{2k+1}) = \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor + 1.$$

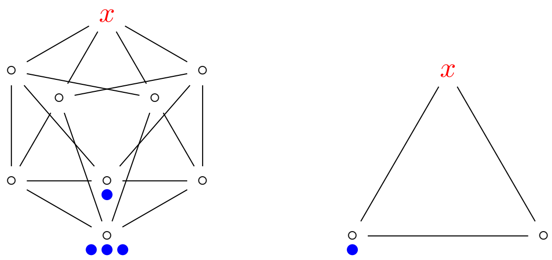
Example 2.14. The figure below illustrates that the rubbling number of the cycle graph C_5 is $\rho(C_5) = 5$. The arrows represent the arrows in the transition digraph of a rubbling sequence that reaches vertex x.



2.4 Graham's Conjecture

The rubbling version of Graham's Conjecture is not true.

Example 2.15. We have $\rho(C_3 \square C_3) = 5 > 4 = 2 \cdot 2 = \rho(C_3)\rho(C_3)$. The figures below show the pebble distributions that prove the lower bounds on the rubbling numbers.



$C_3 \square C_3$ $\qquad$ C_3

x is not reachable

2.5 The t-rubbling number

There is very little known about the t-rubbling number.

Definition 2.16. The *t-rubbling number* of a graph G is the minimum number $\rho_t(G)$ of pebbles in a pebble distribution that makes any vertex reachable with t pebbles, no matter how the pebbles are placed on the vertices.

Proposition 2.17. *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for T, then*

$$\rho_t(T) = t2^{p_1} + \sum_{i=2}^{m} 2^{p_i - 1} - m + 1.$$

Conjecture 2.18. *The t-rubbling numbers of the cycle graphs are*

$$\rho_t(C_{2k+1}) = \frac{2^{k-1}7 + (-1)^k}{3} + (t-1)2^k \quad \text{and} \quad \rho_t(C_{2k}) = t2^k.$$

Question 2.19. *What are the t-rubbling numbers of other simple graph families?*

The following is almost certainly a consequence of the result of [7].

Conjecture 2.20. *There is a t_0 such that the function $t \mapsto \rho_t(G)$ is linear for $t \geq t_0$.*

Question 2.21. *Is there a t_0 that works for all G?*

Question 2.22. *Is there a graph G with diameter d for which $t \mapsto \rho_t(G) - 2^d t$ is not a decreasing function? It is very likely decreasing for t not smaller than the number of vertices.*

The pebbling version of the following is considered in [6].

Definition 2.23. The *t-target rubbling number* of G is the minimum number $\rho(G, t)$ of pebbles in a pebble distribution that makes any goal distribution with t pebbles reachable, no matter how the pebbles are placed on the vertices.

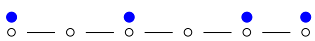
Question 2.24. *Do we have $\rho(G, t) = \rho_t(G)$ for all G and t?*

2.6 Optimal rubbling

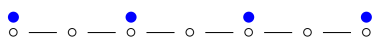
An important tool for finding optimal pebble distributions is *smoothing* where we try to distribute too many pebbles on a single vertex to neighboring vertices while not loosing the reachability of any vertex. Integer programming is also often useful.

Definition 2.25. The optimal rubbling number of a graph G is the minimum number of pebbles $\rho_{\text{opt}}(G)$ in a well-chosen pebble distribution from which any vertex is reachable.

Example 2.26. The figure below shows optimal pebble distributions.



$\rho_{\text{opt}}(P_6) = 4$



$\rho_{\text{opt}}(P_7) = 4$

Now we present the optimal rubbling numbers of some basic graph families.

Proposition 2.27.

1. $\rho_{opt}(P_n) = \lceil \frac{n+1}{2} \rceil$
2. $\rho_{opt}(C_n) = \lceil \frac{n}{2} \rceil$
3. $\rho_{opt}(K_n) = 2$
4. $\rho_{opt}(W_n) = 2$
5. $\rho_{opt}(K_{m,n}) = 3$
6. $\rho_{opt}(Petersen) = 4$

Proposition 2.28. *[12] The optimal rubbling number of the ladder is*

$$\rho_{opt}(P_{3k+r} \Box P_2) = 2k + 1 + \lceil \frac{r}{3} \rceil.$$

Question 2.29. *What are the optimal rubbling numbers $\rho_{opt}(P_n \square P_m)$ and $\rho_{opt}(C_n \square C_m)$?*

Question 2.30. *What is the pebble to vertex ratio in an optimal pebble distribution on the infinite 2-dimensional grids (triangular, rectangular, hexagonal) and on the infinite 3-dimensional rectangular grid?*

Question 2.31. *How does optimal rubbling change if we only allow strict rubbling moves?*

Question 2.32. *Does the optimal rubbling version of Graham's conjecture hold? The answer is most likely yes. It does hold for optimal pebbling.*

2.7 Cover rubbling

The following probably can be shown simply following [20].

Conjecture 2.33. *The cover pebbling and cover rubbling numbers are the same for all graphs. The Stacking Lemma remains true for rubbling.*

2.8 Computational complexity

Proposition 2.34. *The decision problem whether a vertex is reachable from a given configuration is NP-complete.*

Conjecture 2.35. *Deciding whether $\rho(G) \leq k$ is Π_2^P-complete.*

A natural approach for a proof would be to use the corresponding pebbling results.

Question 2.36. *How can we reduce pebbling to rubbling?*

References

[1] Christopher Belford and Nándor Sieben. Rubbling and optimal rubbling of graphs. *Discrete Math.*, 309(10):3436–3446, 2009.

[2] David P. Bunde, Erin W. Chambers, Daniel Cranston, Kevin Milans, and Douglas B. West. Pebbling and optimal pebbling in graphs. *J. Graph Theory*, 57(3):215–238, 2008.

[3] Melody Chan and Anant P. Godbole. Improved pebbling bounds. *Discrete Math.*, 308(11):2301–2306, 2008.

[4] Fan R. K. Chung. Pebbling in hypercubes. *SIAM J. Discrete Math.*, 2(4):467–472, 1989.

[5] Lisa Danz. Optimal t-rubbling of complete m-ary trees. REU project report, University of Minnesota Duluth, Department of Mathematics and Statistics, 2010.

[6] D. S. Herscovici, B. D. Hester, and G. H. Hurlbert. t-pebbling and extensions. *Graphs Combin.*, 29(4):955–975, 2013.

[7] Michael Hoffmann, Jiří Matoušek, Yoshio Okamoto, and Philipp Zumstein. The t-pebbling number is eventually linear in t. *Electron. J. Combin.*, 18(1):Paper 153, 4, 2011.

[8] Glenn Hurlbert. The graph pebbling page. `http://mingus.la.asu.edu/~hurlbert/pebbling/pebb.html`.

[9] Glenn Hurlbert. A survey of graph pebbling. In *Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999)*, volume 139, pages 41–64, 1999.

[10] Glenn Hurlbert. Recent progress in graph pebbling. *Graph Theory Notes N. Y.*, 49:25–37, 2005.

[11] Glenn Hurlbert. Graph pebbling. In Jonathan L. Gross, Jay Yellen, and Ping Zhang, editors, *Handbook of Graph Theory*. CRC Press, 2nd edition, 2014.

[12] Gyula Y. Katona and László Papp. The optimal rubbling number of ladders, prisms and möbius-ladders. http://arxiv.org/abs/1411.0923, 2014.

[13] Gyula Y. Katona and Nándor Sieben. Bounds on the rubbling and optimal rubbling numbers of graphs. *Graphs and Combinatorics*, (to appear).

[14] Kevin Milans and Bryan Clark. The complexity of graph pebbling. *SIAM J. Discrete Math.*, 20(3):769–798 (electronic), 2006.

[15] Lior Pachter, Hunter S. Snevily, and Bill Voxman. On pebbling graphs. In *Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995)*, volume 107, pages 65–80, 1995.

[16] László Papp. Optimal rubbling numbers of graphs (in Hungarian). thesis, Budapest University of Technology and Economics, Department of Computer Science and Information Theory, 2010.

[17] László Papp. Optimal rubbling numbers of prisms and ladders (in Hungarian). preprint, Budapest University of Technology and Economics, Department of Computer Science and Information Theory, 2011.

[18] Nándor Sieben. Census of pebbling numbers. http://www.cefns.nau.edu/~ns46/pebbling/census.html.

[19] Nándor Sieben. The graph rubbling page. http://jan.ucc.nau.edu/ns46/rubblingWeb/Rubbling.html.

[20] Jonas Sjöstrand. The cover pebbling theorem. *Electron. J. Combin.*, 12:Note 22, 5 pp. (electronic), 2005.

A useful homomorphic image of a free abelian by infinite cyclic group

June 10, 2016

Christopher Briggs
Department of Mathematics
Embry-Riddle Aeronautical University

1 Introduction

We introduce a homomorphic image of finitely generated (free abelian)-by-(infinite cyclic) groups with nice computational properties.

2 Background

Throughout, G will be a finitely generated group. Let $\sigma \in \mathrm{Aut}(G)$ be of infinite order.

Definition 1. Let G be a group, $N \triangleleft G$, and $G/N \cong \mathbb{Z}$. THen G is an *infinite cyclic extension* of N. If G/N is generated by t, then t induces an automorphism σ on N. We write $G \cong N \rtimes_\sigma \mathbb{Z}$.

Suppose G is free abelian on $d \geq 2$ generators, $\sigma \in \mathrm{Aut}(G)$ of infinite order, and $\Gamma \cong G \rtimes_\sigma \mathbb{Z}$. The action of σ on G can be represented by a matrix $A \in M_d(\mathbb{Z})$. Since σ is of infinite order, A has an eigenvalue λ which is not of magnitude 1. Letting $\{x_1, \ldots, x_d\}$ be a Jordan basis for A, we have

$$A = \begin{pmatrix} * & * & & 0 \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & \lambda \end{pmatrix}$$

where $x_d^\sigma = \lambda\sigma$. Further, x_d has the following property: if $1 \leq k \leq d$ and $\sigma(x_k) = \sum a_{kj}x_j$, then $\alpha_{kd} \neq 0 \implies k = d$.

We define the homomorphism ϕ as follows. Let $G = \langle g_1, g_2, \ldots, g_d \rangle$, and for each i, write $g_i = \alpha_1^{(i)}x_1 + \cdots + \alpha_d^{(i)}x_d$. Let $\Gamma = G \rtimes_\sigma \mathbb{Z}$, and define $\phi : \Gamma \to \overline{\Gamma}$ by $g_it^k \mapsto (\alpha_d^{(i)})t^k$.

10.5281/zenodo.2579459

We now describe the significance of the homomorphism. Let M be the $\mathbb{Z}[\lambda, \lambda^{-1}]-$module generated by $\{\alpha_d^{(1)}, \ldots, \alpha_d^{(d)}\}$. Since

$$\lambda^d + c_{d-1}\lambda^{d-1} + \cdots + c_1\lambda + 1 = 0,$$

M is a finitely generated $\mathbb{Z}-$module, generated as a module by

$$\{\alpha_d^{(i)}\lambda^j \mid 1 \leq i \leq d,\, 0 \leq j < d\}$$

over $\mathbb{Z}$. Thus $\overline{\Gamma} = \{(\alpha)t^k \mid \alpha \in M, k \in \mathbb{Z}\}$.

Consider the special case $\lambda \geq 2$. Set $a = (\alpha)t^k$, $b = (\beta)t^m \in \bar{\Gamma}$. Define $a < b$ if $k < m$ or ($k = m$ and $\alpha < \beta$).

Lemma 2. *If a, b are as above, $a < b$, and $u \leq v$, then $au < bv$ and $ua < vb$.*

Lemma 3. *If a, b are as above, then either a, b commute or $\langle a, b\rangle$ freely generate a free noncommutative group.*

Lemma 4. *If a, b are as above, then $ab = ba \iff k = m = 0$ or $a = c^k$ where $b = c^m$ for some $c \in \bar{\Gamma}$.*

Taking stock, under the assumption $\gamma \geq 2$ we have found a totally ordered subgroup of a (free abelian)-by-(infinite cyclic) group in which every pair of elements either commutes or freely generates a free semigroup. The next step is to generalize on γ.

Lemma 5. *Suppose $\lambda > 1$. Choose $k \in \mathbb{Z}$ such that $\lambda^k \geq 2$. Then for any pair $a, b \in \bar{\Gamma}$, without loss of generality a^k, b commute or a^k, b freely generate a free semigroup.*

We can generalize our group G to any group which embeds in $GL_n(\mathbb{Z})$.

3 Application to growth

Let G be a finitely generated group, generated by S, $|S| < \infty$. It is standard to assume that S contains the inverses of each of its elements. We define S^n to be products of length at most n in S. Define $\lambda_{G,S} : G \to \mathbb{Z}_{\geq 0}$ by $\lambda(g) = \min_k\{g \in S^k\}$, that is, the length of a minimal length word in S which is equal to g. By convention $\lambda(e) = 0$. We define the growth function depending on G and S by $\gamma_{G,S}(n) = \{g \in G \mid \lambda(g) \leq n\}$. A group has many generating sets, but it turns out that asymptotic growth does not depend on the chosen generating set. That is, if $\gamma_S(n)$ is bounded by a polynomial, then so is $\gamma_T(n)$ (and of the same degree) for any other generating set T of G. Likewise, if $\gamma_S(n)$ is exponential, so is $\gamma_T(n)$ for any other generating set T of G.

The subject of growth in groups introduced independently by Schwarz and Milnor [8, 7]. It is a classic result that G has polynomial growth if and only if G virtually nilpotent [7, 10].

Gromov defined the concept of uniform exponential growth [5]: G has uniform exponential if

$$\inf_S \left(\lim_n \gamma_S(n)^{1/n} \right) > 1$$

where the infimum is taken over all generating sets S of G. That is, G is exponential, and the constant base of exponential growth is bounded away from 1 as one ranges over all generating sets of G. Clearly the condition is stronger than exponential growth. Gromov asked if it was properly stronger, that is, whether there exists a group of exponential but not uniform exponential growth.

Some examples of groups of uniform exponential growth include free group on at least two letters [5], polycyclic groups which are not polynomially bounded [1], and linear groups over a field of any characteristic which are not polynomially bounded [2]. Wilson constructed a counterexample to the general case of Gromov's question [9].

The standard technique of demonstrating uniform exponential growth is to find in boundedly many steps (with the number of steps independent of the chosen generating set) a pair of free generators of a free semigroup. The author uses the homomorphic image described in Section 2, then, to find free semigroups in boundedly many steps [3].

References

[1] R. Alperin, *Uniform exponential growth of polycyclic groups*, Geom. Dedicata **92** (2002) 105–113

[2] E. Breuillard and T. Gelander, *Uniform independence for linear groups*, Invent. Math. **173** (2008) no. 2, 225–263

[3] C. Briggs, *Examples of Uniform Exponential Growth in Algebras*, J. Algebra Appl., DOI: http://dx.doi.org/10.1142/S0219498817502413

[4] R. I. Grigorchuk, *On Milnor's problem of group growth*, Soviet. Math. Dokl. **28** (1983) 23–26

[5] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981) 53–73

[6] M. Gromov, *Structures metriques pour les varietes riemanniennes*, CEDIC, Paris, 1981, MR 85e:53051

[7] J. Milnor, *Growth in finitely generated solvable groups*, J. Diff. Geom. **2** (1968) 447–449

[8] A. S. Schwarz, *A volume invariant of coverings*, Dokl. Ak. Nauk USSR **105** (1955) 32–34

[9] J. Wilson, *On exponential growth and uniformly exponential growth for groups*, Proc. Amer. Math. Soc. **60** (1976) 22–24

[10] J. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Diff. Geom. **2** (1968) 434–446