Introduction to Graph Pebbling and Rubbling

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Abstract

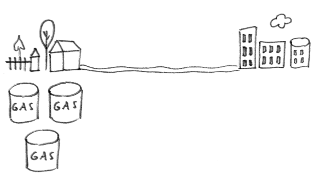
A graph pebbling move removes two pebbles from a vertex of a graph and adds one pebble to an adjacent vertex. In graph rubbling an additional move is allowed that adds a pebble at a vertex after the removal of one pebble each at two adjacent vertices. A vertex is reachable if a pebble can be moved to the vertex using pebbling/rubbling moves. We study the reachability of vertices under different requirements. After a short introduction to pebbling, the known results about rubbling are summarized.

1 Graph pebbling

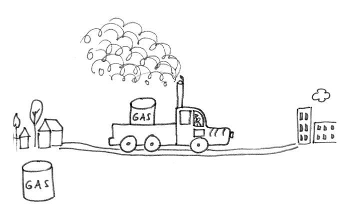
Graph pebbling was originally used as a method for solving a number theory conjecture of Erdős and Lemke. Starting points to the now large literature can be found in [9, 10] and in the more recent [11]. The friendly web page [8] contains many definitions, results, and a large collection of printable articles. A database of pebbling numbers is available at [18].

1.1 Introduction

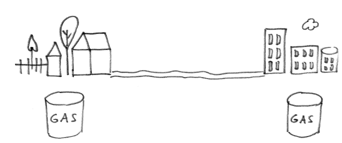
Graph pebbling is a simple model for the transfer of consumable resources. Let's consider a country where old trucks are plentiful but fuel is scarce. Fuel is stored in unit barrels and transporting a barrel from one city to a neighboring city consumes one barrel of fuel. The transfer of a barrel is depicted in the following figures:



before transfer



during transfer



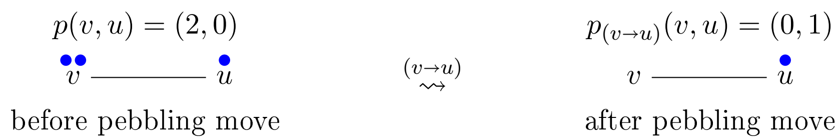
after transfer

Note that the number of fuel barrels has been decreased during the transportation since the fuel was converted to smoke. We now give a more formal definition.

Definition 1.1. Let u and v be adjacent vertices of a graph with vertex set V. A *pebbling move* $(v{\to}u)$ removes two pebbles at v and adds a pebble at u.

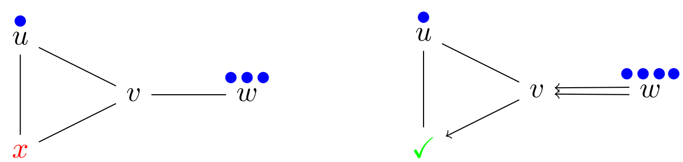
A pebbling move changes the pebble distribution $p : V \to \mathbb{N}$. The resulting pebble distribution $p_{(v\to u)} : V \to \mathbb{N}$ satisfies $p_{(v\to u)}(v, u, *) = (p(v) - 2, p(u) + 1, p(*))$.

Example 1.2. An example of a pebbling move is shown below:



Definition 1.3. The *pebbling number* of a graph G is the minimum number $\pi(G)$ of pebbles in a pebble distribution that makes any vertex reachable, no matter how the pebbles are placed on the vertices.

Example 1.4. The first figure below shows a pebble distribution with 4 pebbles from which vertex x is not reachable. The second figure shows that an additional pebble on vertex w makes vertex x reachable. It is easy to see that every vertex is reachable from any pebble distribution with 5 pebbles, and so the pebbling number of the graph is $\pi(G) = 5$.



We present the pebbling numbers of some basic graph families.

Proposition 1.5.

1. $\pi(P_n) = 2^{n-1}$
2. $\pi(K_n) = n$
3. $\pi(W_n) = n$
4. $\pi(K_{m,n}) = m + n$
5. $\pi(Q_n) = 2^n$
6. $\pi(\text{Petersen}) = 10$

We have some trivial and some more involved bounds on the pebbling number.

Proposition 1.6. *Let G be a graph, d be the diameter of G, n be the number of vertices of G, and $\gamma(G)$ be the domination number of G. Then*

1. $n \leq \pi(G)$
2. $2^d \leq \pi(G)$
3. [3] $\pi(G) \leq (n-d)(2^d - 1) + 1$
4. [3] $\pi(G) \leq (n + 2\gamma(G))2^{d-1} - \gamma(G) + 1$

1.2 The No Cycle Lemma

We formalize the intuition that moving pebbles around in a circle is not helpful for reaching vertices.

Definition 1.7. Let G be a graph with vertex set V. The *transition digraph* of a pebbling sequence (multiset) on G has vertex set V. Every pebbling move $(v\to u)$ adds an arrow $v \longrightarrow u$ to the transition digraph. A pebbling sequence (multiset) is *acyclic* if there is no cycle in the transition digraph.

Lemma 1.8 (No Cycle)**.** *The following are equivalent:*

1. *Vertex v is reachable from the pebble distribution p.*
2. *There is a multiset S of moves such that $p_S \geq 1_{\{v\}}$.*
3. *There is an acyclic multiset R such that $p_R \geq 1_{\{v\}}$.*

4. Vertex v is reachable from p through an acyclic rubbling sequence.

Note that $1_{\{v\}} : V \to \{0,1\}$ is the indicator function. Condition $p_S \geq 1_{\{v\}}$ says that p_S is a pebble distribution ($p_S \geq 0$), and S moved a pebble to vertex v ($p_S(v) \geq 1$).

The No Cycle Lemma shows that we do not have to worry about the ordering of the pebbling moves in a pebbling sequence. If the resulting pebble function is actually a pebble distribution, then an appropriate ordering of the pebbling moves exists.

1.3 Tree graphs and cycle graphs

The *length sequence* of a path partition of a tree is the decreasing finite sequence built from the lengths of the paths in the partition. We order path partitions using the lexicographic order on their length sequences.

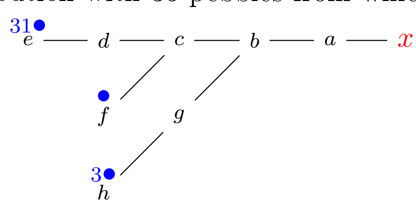
Proposition 1.9 ([4]). *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for a tree T, then*

$$\pi(T) = \sum_{i=1}^{m} 2^{p_i} - m + 1.$$

Example 1.10. The length sequence of the maximum path partition $\{\{x,a,b,c,d,e\},\{b,g,h\},\{c,f\}\}$ of the tree T below is $(5,2,1)$. Hence

$$\pi(T) = (2^5 - 1) + (2^2 - 1) + (2^1 - 1) + 1 = 36.$$

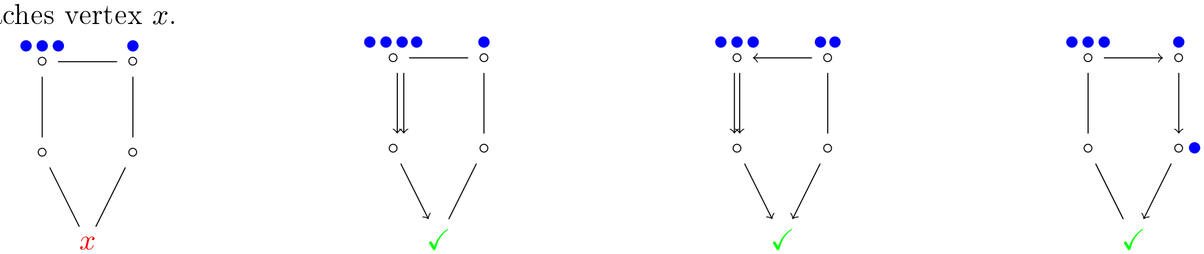
The figure shows a pebble distribution with 35 pebbles from which vertex x is not reachable.



Proposition 1.11 ([15]). *The pebbling number of the cycle graphs are*

$$\pi(C_{2k}) = 2^k \qquad \textit{and} \qquad \pi(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1.$$

Example 1.12. The figure below illustrates that the pebbling number of the cycle graph C_5 is $\pi(C_5) = 5$. The arrows represent the arrows in the transition digraph of a pebbling sequence that reaches vertex x.



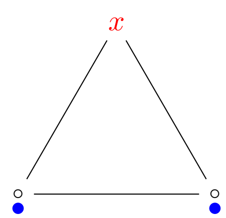
1.4 Graham's Conjecture

The following generated a lot of interest.

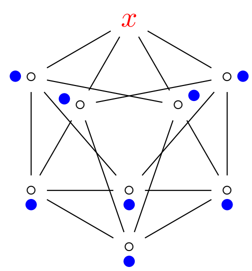
Conjecture 1.13 (Graham)**.** *Every product graph $G\square H$ satisfies $\pi(G\square H) \le \pi(G)\pi(H)$.*

Example 1.14. The figure below shows C_3 and $C_3\square C_3$. Graham's conjecture holds since

$$\pi(C_3\square C_3) = 9 \le 9 = 3 \cdot 3 = \pi(C_3)\pi(C_3).$$



C_3

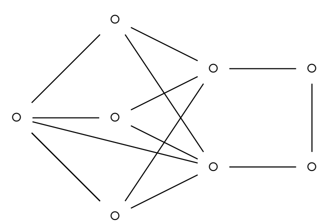


$C_3\square C_3$

x is not reachable

Graham's Conjecture has been verified for many graph families. An important tool in these investigations is the 2-pebbling property. The following graph does not have the 2-pebbling property.

Example 1.15. The Lemke graph L, shown below, is the smallest feasible counterexample for Graham's Conjecture. Unfortunately $L\square L$ is a bit large for computer aided pebbling experimentation. Note that $\pi(G) = 8$.



1.5 The t-pebbling number

In a larger emergency we may want to transfer more than one barrels of fuel to a given location.

Definition 1.16. The *t-pebbling number* of a graph G is the minimum number $\pi_t(G)$ of pebbles in a pebble distribution that makes any vertex reachable with t pebbles, no matter how the pebbles are placed on the vertices.

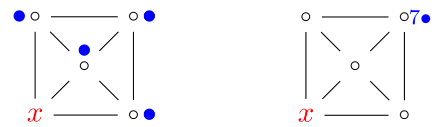
Proposition 1.17 ([4])**.** *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for T, then*

$$\pi_t(T) = t\sum_{i=1}^{m} 2^{p_i} - m + 1.$$

Example 1.18. The t-pebbling number of the wheel graph W_5 is

$$\pi_t(W_5) = \begin{cases} 5, & t = 1 \\ 4t, & t \ge 2. \end{cases}$$

The figures below show the pebble distributions that give the lower bound for $t \in \{1, 2\}$.



$t = 1$ $\qquad$ $t = 2$

Theorem 1.19 ([7]). *There is a t_0 such that the function $t \mapsto \pi_t(G)$ is linear for $t \geq t_0$.*

Question 1.20. *Is it true that $\pi_t(G)$ is the maximum of a few functions that are linear in t, that is, $\pi_t(G) = \max\{a_1 + b_1 t, \ldots, a_k + b_k t\}$?*

1.6 Optimal pebbling

In a well organized country, scientists work hard to store the fuel reserves at the best locations to minimize the size (and the cost) of the reserve.

Definition 1.21. The optimal pebbling number of a graph G is the minimum number of pebbles $\pi_{\text{opt}}(G)$ in a well-chosen pebble distribution from which any vertex is reachable.

Example 1.22. The figure below shows optimal pebble distributions.

$\pi_{\text{opt}}(P_6) = 4$ $\qquad$ $\pi_{\text{opt}}(P_7) = 5$

Now we present the optimal pebbling numbers of some basic graph families.

Proposition 1.23.

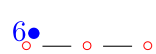
1. [15] $\pi_{opt}(P_n) = \lceil \frac{2n}{3} \rceil$
2. [2] $\pi_{opt}(C_n) = \lceil \frac{2n}{3} \rceil$
3. $\pi_{opt}(K_n) = 2$
4. $\pi_{opt}(W_n) = 2$
5. $\pi_{opt}(K_{m,n}) = 3$
6. $\pi_{opt}(\text{Petersen}) = 4$

1.7 Cover pebbling

Every town might need fuel at the same time in a large emergency.

Definition 1.24. The *cover pebbling number* of a graph G is the minimum number $\pi_{\text{cov}}(G)$ of pebbles to make any vertex reachable at the same time, no matter how the pebbles are distributed on the vertices.

Example 1.25. The cover pebbling number of the P_3 is $\pi_{\text{cov}}(P_3) = 7$. The figure below shows a distribution that proves the lower bound on the cover pebbling number.



Note that all the pebbles are on a single vertex. This in no accident. The following Stacking Theorem is one of the most important results about cover pebbling.

Theorem 1.26 ([20]). *The largest unsolvable pebble configurations contain every pebble on a single vertex.*

1.8 Computational complexity

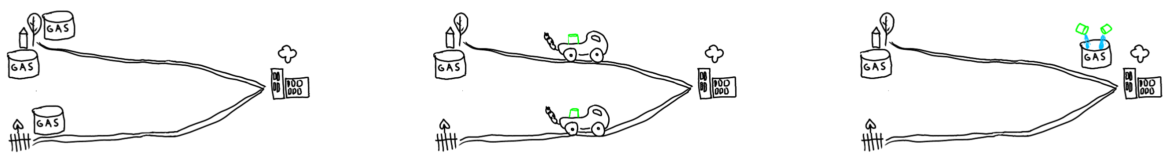
Theorem 1.27 ([14])**.** *Deciding whether a vertex is reachable from a given configuration is NP-complete. Deciding whether $\pi(G) \leq k$ is Π_2^P-complete.*

2 Graph rubbling

The theory of graph rubbling is much less developed than the theory of pebbling. Most of the pebbling results are waiting to be transformed into rubbling results. Rubbling results are sometimes easier and other times are harder than their pebbling versions. All the known results about rubbling are in [1, 5, 16, 17, 13, 12]. An informal guide is available at [19].

2.1 Introduction

In graph rubbling an additional transfer option is available. Smaller trucks now can transport half a barrel of fuel while burning only half a barrel. Two of these can be used to fill a full barrel at the destination town. The transfer of a barrel is depicted in the following figures.



before transfer | during transfer | after transfer

We now give a more formal definition.

Definition 2.1. Let v and w be vertices adjacent to vertex u in a graph with vertex set V. A *strict rubbling move* $(v, w \to u)$ removes one pebble each at vertices v and w, and adds a pebble at u. A *rubbling move* is a pebbling move or a strict rubbling move.

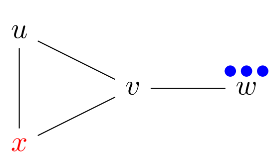
A strict rubbling move changes the pebble distribution $p : V \to \mathbb{N}$. The resulting pebble distribution $p_{(v,w\to u)} : V \to \mathbb{N}$ satisfies $p_{(v,w\to u)}(v, w, u, *) = (p(v) - 1, p(w) - 1, p(u) + 1, p(*))$.

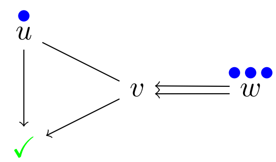
Example 2.2. An example of a strict rubbling move $(v, w \to u)$ on the pebble distribution p is shown below:

$$p(v, w, u) = (1, 1, 0) \quad \overset{(v,w\to u)}{\rightsquigarrow} \quad p_{(v,w\to u)}(v, w, u) = (0, 0, 1)$$

Definition 2.3. The *rubbling number* of a graph G is the minimum number $\rho(G)$ of pebbles in a pebble distribution that makes any vertex reachable, no matter how the pebbles are placed on the vertices.

Example 2.4. The first figure shows a pebble distribution with 3 pebbles from which vertex x is not reachable. The second figure shows that an additional pebble on vertex w makes vertex x reachable. It is easy to see that every vertex is reachable with any pebble distribution with 4 pebbles and so the rubbling number of the graph is $\rho(G) = 4$.





We present the pebbling numbers of some basic graph families.

Proposition 2.5.

1. $\rho(P_n) = 2^{n-1}$
2. $\rho(K_n) = 2$
3. $\rho(W_n) = 4$
4. $\rho(K_{m,n}) = 4$
5. $\rho(Q_n) = 2^n$
6. $\rho(Petersen) = 5$

Note that unlike in the pebbling case, $\rho(G)$ can be smaller than the number of vertices in the graph.

We have the following bounds on the rubbling number.

Proposition 2.6. *If d is the diameter of G, then*

1. $\rho(G) \leq \pi(G)$
2. $2^d \leq \rho(G)$
3. $\rho(G) \leq (n-d+1)(2^{d-1}-1)+2$

Question 2.7. *What property of G makes $\pi(G) = \rho(G)$?*

Question 2.8. *What can we say about the spectrum of rubbling numbers? In particular, are there any gaps in the spectrum?*

2.2 No Cycle Lemma

The No Cycle Lemma holds for rubbling, but we need to adjust the definition of the transition digraph.

Definition 2.9. Let G be a graph with vertex set V. The *transition digraph* of a rubbling sequence (multiset) on G has vertex set V. Every pebbling move $(v, v{\to}u)$ adds the arrows $v \Longrightarrow u$. Every strict rubbling move $(v, w{\to}u)$ adds the arrows $v \longrightarrow u \longleftarrow w$. A rubbling sequence (multiset) is *acyclic* if there is no cycle in the transition digraph.

Lemma 2.10 (No Cycle)**.** *The following are equivalent:*

1. *Vertex v is reachable from the pebble distribution p.*
2. *There is a multiset S of moves such that $p_S \geq 1_{\{v\}}$.*
3. *There is an acyclic multiset R such that $p_R \geq 1_{\{v\}}$.*
4. *Vertex v is reachable from p through an acyclic rubbling sequence.*

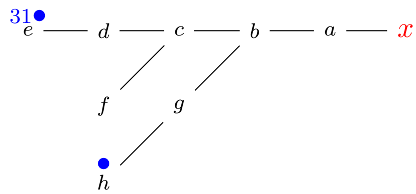
2.3 Tree graphs and cycle graphs

Theorem 2.11. *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for a tree T, then*

$$\rho(T) = 2^{p_1} + \sum_{i=2}^{m} 2^{p_i - 1} - m + 1$$

Example 2.12. The length sequence of the maximum path partition of the tree T below is $(5, 2, 1)$. Hence

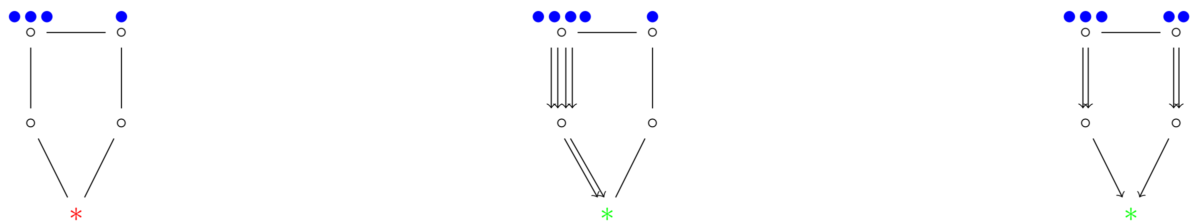
$$\rho(T) = (2^5 - 1) + (2^1 - 1) + (2^0 - 1) + 1 = 33$$



Theorem 2.13. *The rubbling number of the cycle graphs are*

$$\rho(C_{2k}) = 2^k \qquad \text{and} \qquad \rho(C_{2k+1}) = \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor + 1.$$

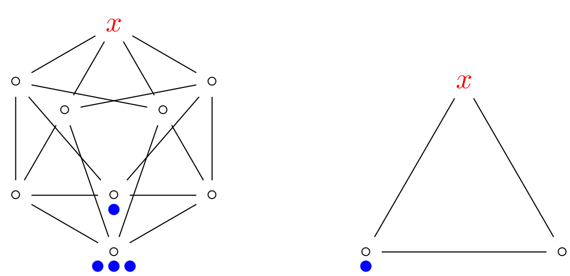
Example 2.14. The figure below illustrates that the rubbling number of the cycle graph C_5 is $\rho(C_5) = 5$. The arrows represent the arrows in the transition digraph of a rubbling sequence that reaches vertex x.



2.4 Graham's Conjecture

The rubbling version of Graham's Conjecture is not true.

Example 2.15. We have $\rho(C_3 \square C_3) = 5 > 4 = 2 \cdot 2 = \rho(C_3)\rho(C_3)$. The figures below show the pebble distributions that prove the lower bounds on the rubbling numbers.



$C_3 \square C_3$ $\qquad$ C_3

x is not reachable

2.5 The t-rubbling number

There is very little known about the t-rubbling number.

Definition 2.16. The *t-rubbling number* of a graph G is the minimum number $\rho_t(G)$ of pebbles in a pebble distribution that makes any vertex reachable with t pebbles, no matter how the pebbles are placed on the vertices.

Proposition 2.17. *If $(p_1, \ldots, p_m)$ is the length sequence of a maximum path partition for T, then*

$$\rho_t(T) = t2^{p_1} + \sum_{i=2}^{m} 2^{p_i - 1} - m + 1.$$

Conjecture 2.18. *The t-rubbling numbers of the cycle graphs are*

$$\rho_t(C_{2k+1}) = \frac{2^{k-1}7 + (-1)^k}{3} + (t-1)2^k \quad \textit{and} \quad \rho_t(C_{2k}) = t2^k.$$

Question 2.19. *What are the t-rubbling numbers of other simple graph families?*

The following is almost certainly a consequence of the result of [7].

Conjecture 2.20. *There is a t_0 such that the function $t \mapsto \rho_t(G)$ is linear for $t \geq t_0$.*

Question 2.21. *Is there a t_0 that works for all G?*

Question 2.22. *Is there a graph G with diameter d for which $t \mapsto \rho_t(G) - 2^d t$ is not a decreasing function? It is very likely decreasing for t not smaller than the number of vertices.*

The pebbling version of the following is considered in [6].

Definition 2.23. The *t-target rubbling number* of G is the minimum number $\rho(G, t)$ of pebbles in a pebble distribution that makes any goal distribution with t pebbles reachable, no matter how the pebbles are placed on the vertices.

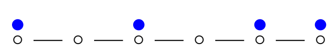
Question 2.24. *Do we have $\rho(G, t) = \rho_t(G)$ for all G and t?*

2.6 Optimal rubbling

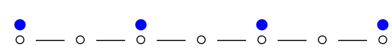
An important tool for finding optimal pebble distributions is *smoothing* where we try to distribute too many pebbles on a single vertex to neighboring vertices while not loosing the reachability of any vertex. Integer programming is also often useful.

Definition 2.25. The optimal rubbling number of a graph G is the minimum number of pebbles $\rho_{\text{opt}}(G)$ in a well-chosen pebble distribution from which any vertex is reachable.

Example 2.26. The figure below shows optimal pebble distributions.



$\rho_{\text{opt}}(P_6) = 4$



$\rho_{\text{opt}}(P_7) = 4$

Now we present the optimal rubbling numbers of some basic graph families.

Proposition 2.27.

1. $\rho_{opt}(P_n) = \lceil \frac{n+1}{2} \rceil$
2. $\rho_{opt}(C_n) = \lceil \frac{n}{2} \rceil$
3. $\rho_{opt}(K_n) = 2$
4. $\rho_{opt}(W_n) = 2$
5. $\rho_{opt}(K_{m,n}) = 3$
6. $\rho_{opt}(Petersen) = 4$

Proposition 2.28. *[12] The optimal rubbling number of the ladder is*

$$\rho_{opt}(P_{3k+r} \Box P_2) = 2k + 1 + \lceil \frac{r}{3} \rceil.$$

Question 2.29. *What are the optimal rubbling numbers* $\rho_{opt}(P_n \Box P_m)$ *and* $\rho_{opt}(C_n \Box C_m)$*?*

Question 2.30. *What is the pebble to vertex ratio in an optimal pebble distribution on the infinite 2-dimensional grids (triangular, rectangular, hexagonal) and on the infinite 3-dimensional rectangular grid?*

Question 2.31. *How does optimal rubbling change if we only allow strict rubbling moves?*

Question 2.32. *Does the optimal rubbling version of Graham's conjecture hold? The answer is most likely yes. It does hold for optimal pebbling.*

2.7 Cover rubbling

The following probably can be shown simply following [20].

Conjecture 2.33. *The cover pebbling and cover rubbling numbers are the same for all graphs. The Stacking Lemma remains true for rubbling.*

2.8 Computational complexity

Proposition 2.34. *The decision problem whether a vertex is reachable from a given configuration is NP-complete.*

Conjecture 2.35. *Deciding whether* $\rho(G) \leq k$ *is* Π_2^P*-complete.*

A natural approach for a proof would be to use the corresponding pebbling results.

Question 2.36. *How can we reduce pebbling to rubbling?*

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