Models of Discrete Conformal Geometry

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Conformal geometry is the study of mappings that preserve angles. The classic examples come from elementary complex analysis: one-to-one analytic mappings between two regions of the complex plane, including common functions like polynomials and exponentials (with domains appropriately restricted). *Discrete* conformal geometry refers to a theory moving results of classical conformal geometry and complex analysis to the discrete setting of graphs.

To do this, we must develop some notion of what discrete conformal mapping should be. We would like to mimic the classical situation as best we can, so let's summarize some features that characterize them:

Angles are preserved. This is our definition, but what could this mean on a graph?

Infinitesimal circles are mapped to infinitesimal circles. This gets a notion of angle in a more useful way. The Cauchy-Riemann Equations from complex analysis say that an analytic mapping locally behaves like a rotation and a dilation, maps which take circles to circles.

Extremal length is preserved. Extremal length is a powerful conformal invariant that translates particularly well to the discrete setting. We will define it formally in Section 2.

Mappings provide a conformal coordinate system on a region. If we have a coordinate system in a region where, for example, we might have a grid of perpendicular axes, then mapping the region conformally should carry the grid to another one with coordinate curves remaining perpendicular. We can think of these coordinates as instructions on how to conformally "straighten" the set.

1 Circle Packing

As we look to discretize these properties, we recognize that "preservation of angles" is a non-starter; we have no notion of angle for a graph. The property that infinitesimal circles map to infinitesimal circles, however, can be made to work.

This gives rise to the idea of a circle packing, which is a collection of circles that meet in externally tangent triples. There is a natural graph associated to

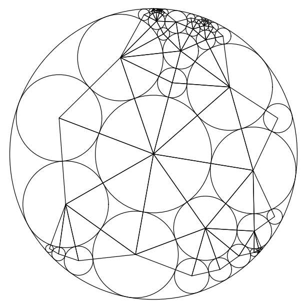

Figure 1: *A circle packing with centers connected to show the tangency graph. All boundary circles are internally tangent to the unit circle, illustrating the discrete Riemann mapping theorem. Drawn with Ken Stephenson's* CirclePack *software.[11]*

any circle packing. Vertices correspond to the circles and two vertices form an edge if their corresponding circles are tangent in the packing. We can see this graph by connecting the centers of the packed circles.

The radii of the circles are positive weights on the vertices. If we change the radii of the circles without changing the tangencies, the packing will look different but the underlying graph is unchanged. This is our notion of a discrete analytic function, borrowing from our working property of conformal mapping but removing the discrete-unfriendly word "infinitesimal." We should expect to see this "infinitesimal" emerge as some kind of limit in our discrete model. We will illustrate exactly that by first considering one of the most celebrated theorems in classical analysis.

Theorem 1 (Riemann Mapping Theorem) *Any open simply connected proper subset of the complex plane can be mapped to the open unit disk by a bijective analytic map. This map is unique up to automorphisms of the disk.*

This is amazing. Conformality seems to be a strong condition, but it is sufficient to take any pathological simply connected monstrosity nicely onto the disk. Moreover, the condition is essentially unique after accounting for the three-parameter family of maps from the disk to itself.

Here is the discrete analog:

Theorem 2 (Discrete Riemann Mapping Theorem) *Any finite triangulation of a disk realizes a circle packing whose boundary circles are internally tangent to the unit circle.*

This theorem was established as a linchpin for discrete conformal geometry by William Thurston (although it was actually proved much earlier, c.f. [2], [8], [13]). To see how the two versions tie to together, consider a bounded simply connect region R in the plane. Cover the plane with the "penny packing" formed by a packing of identical circles with each circle tangent to exactly six neighbors. Carve out a portion of this packing including every circle whose interior contains at least one point in R. If the radius of the packed circles is small relative to R, then the region filled by this collection of circles should approximate R.

This is a circle packing. By the Discrete Riemann Mapping Theorem, the radii of these currently identical circles may be adjusted without altering the underlying combinatorics so that the boundary circles lie internally tangent to the unit circle, and indeed algorithms exist to approximate these radii. We impose uniqueness by, say, forcing some chosen circle to map to the origin and another to lie on the positive real axis. This discrete analytic mapping takes a point in R that is a center of a circle to a point inside the unit disk, which is the center of its corresponding circle. We can extend continuously to points that are not centers by using barycentric coordinates. Now we see how to get that limiting process we wanted. Repeat this procedure on penny packings of smaller and smaller radius (with consistent normalizations), giving better and better approximations to the region R.

Theorem 3 *The sequence of discrete analytic mappings described above converges uniformly on compact sets to the Riemann Mapping of R to the disk, with the corresponding normalizations.*

The first version of this theorem was proved by Rodin and Sullivan in [9] and has since been extended in various ways. He and Schramm [7] proved a version powerful enough to count as an alternate proof of the Riemann Mapping Theorem itself. (We are overlooking a long story about the precise statements of these results, among other things. See [12] and the references therein.)

2 Extremal Length

Of significant interest is determining which sets can be mapped to one another by a conformal mapping, in which case we call the sets *conformally equivalent*. The Riemann Mapping Theorem essentially says that all proper open simply connected subsets of the plane are conformally equivalent. The story gets more interesting with quadrilaterals, where we further require that the mapping to carry vertices to vertices. A *conformal invariant* is a property that must be shared by two conformally equivalent sets.

Define a *topological quadrilateral* to be a simply connected domain with four distinguished vertices that divide the boundary into four arcs. Choose one non-intersecting pair of these arcs to be the "top" and "bottom."

Extremal length is a conformal invariant developed by Lars Ahlfors [1] and has emerged as a powerful tool in conformal geometry. We will skip to the punchline to motivate this tool. The Riemann Mapping Theorem tells us that any open simply connected region in the complex plane can be mapped conformally to a disk. Consider this domain as a topological quadrilateral we wish to map to a rectangle with vertices mapping to corresponding vertices. The Riemann mapping is unique up to three parameters of automorphisms, so specifying four points costs us a degree of freedom. We pay this debt in the aspect ratio (length divided by width) of the image rectangle, which is forced upon us and is a conformal invariant of the quadrilateral. This will turn out to be the extremal length, although we actually define it in terms of path families.

Define a *metric* to be a positive function on the quadrilateral Q and its *area* to be $\int\int_Q m^2\ dA$. We restrict ourselves to the set Λ of metrics with positive area. If γ is a curve lying in Q, its length with respect to a metric m is $\int_\gamma m\ d\gamma$. Finally, let Γ be the set of all rectifiable curves in Q connecting the top arc to the bottom. Extremal length is defined as follows.

$$\mathrm{EL}(Q) = \sup_{m\in\Lambda} \frac{\inf_{\gamma\in\Gamma}(\int_\gamma m\ |dz|)^2}{\int\int m^2\ dA}$$

The infimum in the numerator is finding the shortest (squared) length path connecting opposite sides with respect to a metric. The denominator is the area. Thinking of rectangles, this is just

$$\frac{\text{length}^2}{\text{area}} = \frac{\text{length}^2}{\text{length}\times\text{width}} = \frac{\text{length}}{\text{width}},$$

which is the aspect ratio. It thus makes some sense to define the quantity inside the supremum as the aspect ratio of the metric. The supremum then searches through all metrics on the quadrilateral for the one with the largest aspect ratio. It turns out that this metric exists and is indeed the norm of the derivative of the Riemann mapping onto a rectangle. Note that the ratio of length over width is automatically scale invariant so we often restrict to metrics normalized to area one. To see that this is a conformal invariant, consider another quadrilateral Q' that is conformally equivalent to Q. By definition, there is some conformal mapping ρ from Q to Q', and $|\rho'|$ will necessarily by included in Λ. In other words, the supremum automatically sifts through conformal equivalences. We should note that the definition of extremal length puts no restrictions on the set Γ of curves and there are lots of reasons to study other curve families, but curves connecting opposite sides of quadrilaterals are sufficient for our purposes. We also point out that the choice of which pair of arcs are the top and bottom does matter, but choosing the other pair simply reciprocates the aspect ratio (i.e., switches the roles of length and width).

This definition is easy to port to a discrete setting − we just change the regions to graphs, the curves to vertex paths, and the integrals to sums.

$$\mathrm{EL}(G) = \sup_{\rho\in\hat{\Lambda}} \frac{\inf_{\gamma\in\hat{\Gamma}}(\sum_{v\in\gamma} m(v))^2}{\sum_{v\in V} m(v)^2}$$

G is a combinatorial quadrilateral, defined as a planar triangulation with boundary divided into four vertex paths, with two disjoint arcs designated the

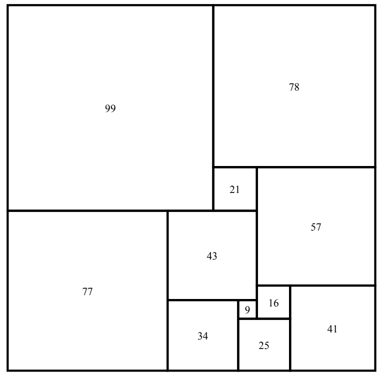

Figure 2: *A square tiling of a* 176×177 *rectangle. The extremal length of the associated discrete quadrilateral is thus* $\frac{177}{176}$. *This is one of the first examples of a* perfect tiling *in which no two squares are congruent [4].*

"top" and "bottom," just as in the classical case. $\hat{\Lambda}$ is the set of positive functions m on the vertices V of G with area $\sum_{v \in V} m(v)^2 > 0$, and $\hat{\Gamma}$ is the set of vertex paths in G connecting the top of G to the bottom.

The proof that extremal length exists is not too hard. Consider a vector space whose basis is the set of vertices of G. Then a metric, being an assignment of values to each vertex, is an element of this finite-dimensional vector space. Since extremal length is scale invariant, we may restrict to unit area metrics, i.e. points on the unit sphere in our vector space. One need only verify that the aspect ratio is a continuous function, so we are looking for a maximum of a continuous function on a compact set, which always exists. Uniqueness can be proved from convexity of the sphere. See [5] for details.

So the metric exists and is unique, but what is it? It turns out that if we use the graph as a tangency graph for squares of side length $m(v)$, then those squares will fit together in a perfect rectangle. Like the classical case, we have no control over the aspect ratio of the rectangle, which will be the extremal length. These square tilings of rectangles are similar to circle packings except that the non-smoothness of squares requires us to allow some degenerate cases.

An interesting schism in the theory now emerges. If we refine the graph as we did with circle packings, the discrete mappings induced by square tilings will not generally converge to the Riemann mapping of a region to a rectangle. Extremal length captures some aspects of conformality and circle packings capture others. Discretization has finally cost us.

3 Electric Networks and Other Models

We now consider how Riemann himself thought about conformal mapping. Suppose we have a topological cylinder made from some conductive metal. Connect a battery to the two boundary circles to draw a current from one boundary circle to the other. The electrons will move between these circles along flow lines while the curves of equal potential will necessarily be orthogonal to these flow lines. But then these flow lines and equipotential curves give conformal coordinates that map the topological cylinder to a straight rectangular one, which was our final feature of conformal mappings. The resistance of this is system – which is clearly predetermined by the physical interpretation – will be captured by the height of the rectangular cylinder. This is extremal length again.

To discretize this idea, consider the edges of a graph to be wires with unit resistance. If we draw a current between the boundary components, then there will be some effective resistance between the components. This can sometimes be computed using Kirchoff's Laws. This approach brings us back to square tiling, although here the squares are represented by edges rather than vertices. See [3] for a nice exposition.

A similar model is random walk. Doyle and Snell [6] show that this is equivalent to the electric network model by showing that they both arise from discrete harmonic functions (i.e., functions whose values are the averages of the values of neighboring points). These functions are uniquely determined by boundary conditions, so the two systems must yield the same functions. The classical analog is Brownian motion.

Other models can be obtained from different kinds of packings. We have studied circles and squares, but many similar results can be obtained by packing more general shapes. We may also relax the tangency condition (necessary if we start varying the shapes of tiles). Most models tend to behave similarly if the underlying triangulations have bounded degree, meaning there is an $N > 0$ such that every vertex has at most N adjacent vertices. This condition can usually translate into a bound on how the discrete function deviates from a conformal mapping, inviting techniques of *quasiconformal analysis* to force nice limiting behaviors. Without this condition, however, quasiconformal methods fail and the peculiarities of the models reveal themselves.

4 Onward

All of our characterizations of conformal mapping have discrete analogs, but different characterizations required meaningfully different models. Discrete conformal geometry has grown into a large field with powerful theoretical and computational results. As mathematicians push these results further, however, the gaps between the different models become more interesting.

For example, let's tweak our definition of extremal length by assigning the metric to the edges of the graph instead of to the vertices. Everything else about the definition as maximal aspect ratio remains the same. It does not seem at first blush that this should change the story much, but it turns out that the switch from vertices to edges recaptures the circle packing versus square tiling dichotomy. See [14].

Many questions remain open about each of these models, and many more are available if we want to look deeply into the differences among them. The insights we gain into how these models do and do not capture classical behaviors have potential to lead to more effective use of these tools as we pursue questions in both settings.

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