Higher Galois for Segal Topos and Natural Phenomena

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Abstract

In [TV1], Toen and Vezzosi show that $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(T, \mathcal{X})$ is a Segal groupoid, for T a Segal topos, $\mathcal{X} = \mathrm{Loc}(X)$ the Segal category of locally constant stacks on a CW complex X. Taking the realization of such a groupoid as in [HS] defines a pro-object $H_T = |\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(T, -)|$ that is defined to be the homotopy shape of the topos T ([TV1]). What we do instead is fix $\mathcal{X}$, any Segal topos, and let T vary, and use the fact that $\mathbb{R}\underline{\mathrm{Hom}}^{*}_{Lex}(\mathcal{X}, T) = \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(T, \mathcal{X})$ is a fundamental ∞-groupoid. We then prove that $\mathcal{X}$ is a localization of the Segal category of local systems on $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(T, \mathcal{X})$, in the spirit of [Hoy] where it is proved, morally, that local systems on H_T are equivalent to T itself. We provide one application of this formalism, regarding the Segal topos $\mathcal{X} = \mathrm{dSt}(k)$ of derived stacks, for k a commutative ring, as corresponding to manifestations of natural laws, themselves modeled by simplicial algebras, objects of sk-CAlg ([TV2], [TV3], [TV4], [T]).

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1 Introduction

From the perspective of shape theory, one can define the shape of a Segal topos T, as Toen and Vezzosi did in [TV1], as being defined by $H_T = |\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(T,-)|$. Here $|\ \ |$ denotes the realization of a Segal category as in [HS], left adjoint of the fundamental groupoid functor Π_∞. In [TV1] it is proven that for X a CW complex, we have for $T = \mathrm{Loc}(*) = \mathrm{Top}$, $X \simeq |\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(\mathrm{Top}, \mathrm{Loc}(X))|$, or equivalently $\Pi_\infty(X) \simeq \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(\mathrm{Top}, \mathrm{Loc}(X))$. In SGA 1 however, Grothendieck suggests that the category of fiber functors from $\mathcal{X}$ to T, $\mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X},T) \simeq \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(T,\mathcal{X})$, could be called a fundamental groupoid. This can be made precise, as we will do in this paper, but this won't be the fundamental ∞-groupoid of $\mathcal{X}$ itself, rather it will be the fundamental ∞-groupoid of left exact functors on $\mathcal{X}$ instead. Indeed recall that for T and $\mathcal{X}$ two Segal topos, or more generally, two topos, a geometric morphism f from T to $\mathcal{X}$ consists of a pair of adjoint functors, $f_* : T \rightarrow \mathcal{X}$ and $f^* : \mathcal{X} \rightarrow T$, $f^* \dashv f_*$, f^* left exact. Thus if SeT denotes the category of Segal topos, we adopt the notation $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T,\mathcal{X})$ for $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}(T,\mathcal{X})$, which, as we will see, is equivalent to $\mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X},T) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T,\mathcal{X})^{\mathrm{op}}$ depending on whether we fix our attention on left adjoints, or right adjoints. Now it turns out fiber functors are left exact, and working with Segal topos it is therefore natural, in the spirit of Grothendieck, to have the groupoid of fiber functors in the Segal setting as being given by $\mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X},T)$, sub-Segal category of $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{X},T) = \widehat{\mathcal{X}^{\mathrm{op}}}$, relative to T of course, which is implied here. Hence we adopt the notation $\mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X},T) = \Pi_{\infty,T}\widehat{\mathcal{X}^{\mathrm{op}}}$. We prove, using Toen's work in [TV1], that this is again a Segal topos for any T. Considering local systems $i_T = \mathbb{R}\underline{\mathrm{Hom}}(-,T)$ on it, we show $\mathcal{X}$ is a localization of $i_T\Pi_{\infty,T}\widehat{\mathcal{X}^{\mathrm{op}}}$. If we apply this in particular to a description of all natural phenomena, one can envision that laws of nature be modeled by simplicial algebras, objects in sk-CAlg for k a commutative ring, realized via derived stacks, the collection of which is a Segal topos $\mathrm{dSt}(k)$, to which we can apply all this formalism. In particular $\mathcal{X} = \mathrm{dSt}(k)$ being a Segal category, $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(\mathcal{X},\mathcal{X}) = \mathcal{U}$ is another Segal category, and $i_{\mathcal{X}}\mathcal{U} = \mathbb{R}\underline{\mathrm{Hom}}(\mathcal{U},\mathcal{X})$ is yet another Segal category $\mathcal{V}$, whose localization gives back $\mathcal{X}$. We leave the reader to draw his own conclusions from such a result; mathematically it appears as nothing more than a simple exercise. Its interpretation however, depending on one's own philosophical convictions, is very deep. One advantage of using stacks to model algebraic realizations of natural laws is that one can completely bypass the use of "target spaces" and "fields" living on such spaces, whose dynamics is given by equations of motion derived from an independent "Lagrangian", notwithstanding the fact that such target spaces for the most part would be obtained from compactification(s). Here dynamics is dictated by coherence conditions inherent in the definition of stacks themselves, and in the fact that all of them combined form a Segal topos, with a basic modus operandi following some basic rules encoded in simplicial algebras. This has the advantage of repackaging all of Physics in a purely Algebro-Geometric object such as a Segal Topos, in the spirit of Kontsevich's take on the Mirror Symmetry problem by introducing the Homological Mirror Symmetry formalism ([K]).

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2 Grothendieck's take on Galois Theory

We briefly remind the reader of the following fundamental result of Galois Theory: for K a finite Galois extension of a field k, $G = \mathrm{Gal}(K/k)$, we have a one-to-one correspondence between subfield extensions $k \subset E \subset K$ and subgroups $H \subset G$, given by $E \mapsto \mathrm{Gal}(K/E)$ and $H \mapsto K^H$. It was Grothendieck's idea in SGA 1 to categorize this result, as clearly recounted in [D], by first using the well-known fact that there is a one-to-one correspondence between conjugacy classes of subgroups of G and isomorphism classes of transitive G-sets, whose category we denote tSets^G, and by regarding field extensions as a category $\mathcal{C}^{\mathrm{op}}$ whose objects are fields, k then becoming the terminal object of $\mathcal{C}$. One then moves into the categorical realm. An object A of $\mathcal{C}$ being fixed, and under the assumption that for all objects X of $\mathcal{C}$, there is a morphism $A \to X$, which is further a strict epi, meaning the joint coequalizer of all the parallel pairs that it coequalizes, if we assume we have a notion of categorical quotient $A \to A/H$, preserved by $\mathrm{Hom}_{\mathcal{C}}(A, -) = [A, -]$, and finally if we assume that $\mathrm{End}(A) = \mathrm{Aut}(A)$, then under those assumptions we have an adjoint equivalence:

$$[A, -] : \mathcal{C} \rightleftarrows \mathrm{tSets}^G : A \times_G - \tag{1}$$

thereby providing an abstraction of the classical Galois statement. Grothendieck's idea then was to observe that $[A, -]$ being a map from $\mathcal{C}$ to Set, one may as well start from a fiber functor $F : \mathcal{C} \to$ fSets, left exact among other things, and under mild conditions on $\mathcal{C}$ as well as F itself, one obtains a generalization of the above result ([G], [D]). One would first show that F is pro-corepresentable, $F = [P, -]$, from which one would make a transition from the result given in (1) to the following adjoint equivalence:

$$F = \mathrm{Hom}_{\mathcal{C}}(P, -) = [P, -] : \mathcal{C} \rightleftarrows \mathrm{fSets}^{\pi} : P \times_{\pi} -$$

$\pi = \mathrm{Aut}(P)^{\mathrm{op}}$ a profinite group. Grothendieck then observed that this depended of course on F and that a way to make this independent of the choice of a fiber functor was to consider $\Gamma = \{F : \mathcal{C} \to \mathrm{fSets}\}$, which he argued is a groupoid. Considering local systems on Γ, meaning introducing $i = \mathrm{Hom}_{\mathcal{C}}(-, \mathrm{fSets})$, he then proved in a few lines that $i\Gamma \simeq \mathcal{C}$.

3 Grothendieck's Galois Theory for Segal Topos

We now promote $\mathcal{C}$ and fSets to the status of Segal topos as introduced in [TV1]. Thus our base category $\mathcal{C}$ becomes a fixed Segal category $\mathcal{X}$, and we consider functors valued in any Segal topos T. Fiber functors in the Segal topos setting would be represented by left exact left adjoints. This justifies the definition:

$$\mathbb{R}\underline{\mathrm{Hom}}^{*}_{\mathrm{SeT}}(\mathcal{X}, T) = \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$$

as given in [TV1] where $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}$ is denoted $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{geom}}$ instead. Recall from the same paper that SePC, the category of Segal pre-categories, is a symmetric monoidal model category ([Ho]) with the direct product as monoidal product, hence Ho(SePC) would have an internal Hom object ([Ho]) denoted $\mathbb{R}\underline{\mathrm{Hom}}(A, B) \in \mathrm{Ho(SePC)}$ for $A, B \in \mathrm{Ho(SePC)}$, where $\mathbb{R}\underline{\mathrm{Hom}}(A, B) \cong \underline{\mathrm{Hom}}(A, RB)$, $\underline{\mathrm{Hom}}$ the internal Hom object in SePC. However, we also have an anti-equivalence $\mathbb{R}\underline{\mathrm{Hom}}^{*}_{\mathrm{SeT}}(\mathcal{X}, T) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \mathcal{X})^{\mathrm{op}}$. That this is an anti-equivalence follows from the commutative diagram below; for $\mathcal{X}$ and $\mathcal{Y}$ two Segal topos, $f, g : \mathcal{X} \to \mathcal{Y}$ two geometric morphisms,

$\alpha : f^* \Rightarrow g^*$, $x \in \mathcal{X}_0$ and $y \in \mathcal{Y}_0$, there corresponds $\beta : g_* \Rightarrow f_*$ as in:

$$\begin{tikzcd}
\mathcal{X}(g^*y, x) \arrow[r, "\cong"] \arrow[d, "\alpha_y^*"'] & \mathcal{Y}(y, g_*x) \arrow[d, "\beta_{x,*}"] \\
\mathcal{X}(f^*y, x) \arrow[r, "\cong"'] & \mathcal{Y}(y, f_*x)
\end{tikzcd}$$

Hence we can also define $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$ as:

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X}) = \mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \mathcal{X})^{\mathrm{op}}$$

Theorem 3.1. For $\mathcal{X}$ and T two Segal topos, $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$ is a Segal groupoid.

Proof. $\mathcal{X}$ being a Segal topos, by definition there is a fully faithful map $i : \mathcal{X} \to \hat{B} = \mathbb{R}\underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathrm{Top})$, B a Segal Category, $\mathrm{Top} = \mathrm{Loc}(*) = L(\mathrm{sSet})$. This map being fully faithful, which we abbreviate by ff, it induces a ff map $\mathbb{R}\underline{\mathrm{Hom}}(T, \mathcal{X}) \to \mathbb{R}\underline{\mathrm{Hom}}(T, \hat{B})$:

$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{Hom}}(T, \mathcal{X}) \arrow[r, "\mathrm{ff}"] & \mathbb{R}\underline{\mathrm{Hom}}(T, \hat{B}) \\
\mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \mathcal{X}) \arrow[u, "\mathrm{ff}"] \arrow[r, dashed, "\mathrm{ff}"'] \arrow[ur, dotted, "\mathrm{ff}"] & \mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \hat{B}) \arrow[u, "\mathrm{ff}"']
\end{tikzcd}$$

The diagonal map is fully faithful by composition, and the bottom map is so as right adjoints with a left exact left adjoint map to right adjoints with a left exact left adjoint as one can check. Finally using the adjunction formula $\mathbb{R}\underline{\mathrm{Hom}}(A \times B, C) \simeq \mathbb{R}\underline{\mathrm{Hom}}(A, \mathbb{R}\underline{\mathrm{Hom}}(B, C))$ of [TV1] limited to right adjoints with a left exact left adjoint:

$$\begin{aligned}
\mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \hat{B}) &= \mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \mathbb{R}\underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathrm{Top})) \\
&\simeq \mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T \times^{\mathbb{L}} B^{\mathrm{op}}, \mathrm{Top}) \\
&= \mathbb{R}\underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \mathrm{Top})) \\
&= \mathbb{R}\underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathrm{Top})^{\mathrm{op}})
\end{aligned}$$

and then we use the fact that $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathrm{Top})$ is a Segal groupoid as proved in [TV1], so that $\mathbb{R}\underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathrm{Top})^{\mathrm{op}})$ itself is a Segal groupoid, hence so is $\mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \hat{B})$, and $\mathbb{R}\underline{\mathrm{Hom}}_{*,\mathrm{SeT}}(T, \mathcal{X})$ faithfully maps into it, so is a Segal groupoid, or equivalently, $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$ is a Segal groupoid. □

Regarding notations, since $\widehat{\mathcal{X}^{\mathrm{op}}} = \mathbb{R}\underline{\mathrm{Hom}}(\mathcal{X}, T)$ relative to T, and that $\mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X}, T)$ is a full-sub Segal category thereof ([TV1]), then we will write $\Pi_{\infty,T}\widehat{\mathcal{X}^{\mathrm{op}}}$ for $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$. This then defines a functor, for any Segal topos T:

$$\begin{aligned} \Pi^{\wedge}_{\infty,T} : \mathrm{SeT} &\to \mathrm{SeGpd} \\ \mathcal{X} &\mapsto \Pi_{\infty,T}\widehat{\mathcal{X}^{\mathrm{op}}} \end{aligned}$$

where SeGpd denotes the category of Segal groupoids. We now define:

$$\begin{aligned} i_T : \mathrm{SeGpd} &\to \mathrm{SeCat} \\ A &\mapsto i_T(A) = \mathbb{R}\underline{\mathrm{Hom}}(A, T) \end{aligned}$$

Theorem 3.2. For $T \in \mathrm{SeT}$, $i_T = \mathbb{R}\underline{\mathrm{Hom}}(-, T)$, then:

$$i_T : \mathrm{SeGpd} \rightleftarrows \mathrm{SeT} : \Pi^{\wedge}_{\infty,T}$$

$$i_T \dashv \Pi^{\wedge}_{\infty,T}$$

It follows that for all Segal topos $\mathcal{X}$ we have a unique counit map:

$$i_T\Pi_{\infty,T}\widehat{\mathcal{X}^{\mathrm{op}}} \to \mathcal{X}$$

Proof. It suffices to write, for $A \in \mathrm{SeGpd}$, using the adjunction formula for left exact left adjoints:

$$\begin{aligned} \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(i_T A, \mathcal{X}) &= \mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X}, i_T A) \\ &= \mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X}, \mathbb{R}\underline{\mathrm{Hom}}(A, T)) \\ &\simeq \mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X} \times^{\mathbb{L}} A, T) \\ &\simeq \mathbb{R}\underline{\mathrm{Hom}}(A, \mathbb{R}\underline{\mathrm{Hom}}^*_{\mathrm{SeT}}(\mathcal{X}, T)) \\ &= \mathbb{R}\underline{\mathrm{Hom}}(A, \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})) \\ &= \mathbb{R}\underline{\mathrm{Hom}}(A, \Pi_{\infty,T}\widehat{\mathcal{X}^{\mathrm{op}}}) \end{aligned}$$

where we have chosen to define geometric morphisms by their left adjoints. □

Note that this is functorial in T: a geometric morphism $u : T \to T'$ induces a morphism $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X}) \leftarrow \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T', \mathcal{X})$ ([MM]), hence a morphism $\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X}), T) \to \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T', \mathcal{X}), T)$, so we consider:

$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X}), T) \arrow[r] \arrow[dr, dotted] & \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T', \mathcal{X}), T) \arrow[d] \\
 & \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T', \mathcal{X}), T')
\end{tikzcd}$$

hence giving a map $i_T \Pi_{\infty,T} \widehat{\mathcal{X}^{\mathrm{op}}} \to i_{T'} \Pi_{\infty,T'} \widehat{\mathcal{X}^{\mathrm{op}}}$, giving rise to a colimit diagram:

$$\begin{tikzcd}
i_{\mathrm{Top}} \Pi_{\infty,\mathrm{Top}} \widehat{\mathcal{X}^{\mathrm{op}}} = i \Pi_\infty \widehat{\mathcal{X}^{\mathrm{op}}} \arrow[d, equal] & & \\
\mathrm{colim}_T \, i_T \Pi_{\infty,T} \widehat{\mathcal{X}^{\mathrm{op}}} \arrow[rr, dashed] & & \mathcal{X} \\
 & i_{T'} \Pi_{\infty,T'} \widehat{\mathcal{X}^{\mathrm{op}}} \arrow[ul] \arrow[ur] & \\
 & i_T \Pi_{\infty,T} \widehat{\mathcal{X}^{\mathrm{op}}} \arrow[uul] \arrow[u] \arrow[uur] &
\end{tikzcd}$$

We now argue that $\mathcal{X}$ is a localization of $i_T \Pi_{\infty,T} \widehat{\mathcal{X}^{\mathrm{op}}}$ for all Segal topos T. Had we fixed T and let $\mathcal{X}$ vary, taking the realization of the Segal groupoid $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$, we would have found that we have a fully faithful embedding as in [Hoy]. However our interest was not in Shape Theory but in Grothendieck's interpretation of Galois Theory only, with a special emphasis on fiber functors, the collection of which is the Segal groupoid $\Pi_{\infty,T} \widehat{\mathcal{X}^{\mathrm{op}}}$.

Theorem 3.3. For all Segal Topos T, a fixed Segal topos $\mathcal{X}$ is a localization $\mathcal{X} = \ell(i_T \Pi_{\infty,T} \widehat{\mathcal{X}^{\mathrm{op}}})$.

Proof. In writing $i_T = \mathbb{R}\underline{\mathrm{Hom}}(-, T)$, we place ourselves in Ho(SePC), so if we write $i_T \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$, the object $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$ is seen as an object in Ho(SePC), and $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X})$ being a Segal groupoid, in the homotopy category of Segal categories, all its objects are isomorphic, so it suffices to consider one of them. Hence:

$$i_T \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X}) \cong \{g^* : \mathcal{X} \xrightarrow{Lex} T\} \times T$$

where Lex stands for left exact. It suffices that we pick one such functor. Consider $g^* = \mathrm{Hom}_{\mathcal{X}}(g, -)$, and more precisely, consider $g = 0_{\mathcal{X}}$, the initial object of $\mathcal{X}$. For all $f \in \mathcal{X}$, $\mathrm{Hom}_{\mathcal{X}}(g, f) \in \mathrm{sSet}$, and we make this T-valued by using the fact that T a Segal topos is in particular a Segal category, or a bisimplicial set as defined in [GJ], where it is shown that bisimplicial sets are tensored over simplicial sets. Hence we have a functor into T by considering $\mathrm{Hom}_{\mathcal{X}}(0_{\mathcal{X}}, -) \otimes 0_T$ for instance. Then with such a choice:

$$i_T \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(T, \mathcal{X}) \simeq \mathcal{X}_{0/} \times T \simeq \mathcal{X} \times T$$

which projects down to $\mathcal{X}$, a left exact map, and $\mathcal{X}$ itself has a fully faithful embedding into $\mathcal{X} \times T$, hence we have a localization as claimed. $\square$

4 Natural Phenomena

We regard natural phenomena as following some basic rules, or fundamental laws, encoded in algebras. Coherent manifestations of such laws will be modeled by stacks valued in simplicial sets, and according to the philosophy of derived algebraic geometry, it is natural then to consider simplicial algebras. Hence we start with a commutative ring k, we let k-Mod be the category of k-modules, commutative monoids of which form sk-CAlg the category of simplicial k-algebras. Its opposite category dk-Aff $= L(\mathrm{s}k\text{-CAlg})^{\mathrm{op}}$ is referred to as the Segal category of derived affine stacks ([TV2], [TV3], [TV4], [T]). We put the ffqc topology on this Segal category. The category $\mathcal{X} = \mathrm{dSt}(k)$ of affine stacks is a localization of the Segal category of pre-stacks $\widehat{\mathrm{d}k\text{-Aff}} = \mathbb{R}\underline{\mathrm{Hom}}(\mathrm{d}k\text{-Aff}^{\mathrm{op}}, \mathrm{Top})$. The former is a Segal topos ([T]), hence all the formalism above applies to dSt(k). In particular one can write:

$$\mathcal{X} = \ell(i_{\mathcal{X}} \Pi_{\infty,\mathcal{X}} \widehat{\mathcal{X}^{\mathrm{op}}}) = \ell(\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(\mathcal{X}, \mathcal{X}), \mathcal{X}))$$

$\mathcal{X}$ being a Segal category, we know there is a Segal category $\mathcal{U}$ such that $\mathcal{U} = \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(\mathcal{X}, \mathcal{X})$. This means that morphisms between stacks in $\mathcal{X}$ are in $\mathcal{U}$. This latter being a Segal category, $\mathbb{R}\underline{\mathrm{Hom}}(U, \mathcal{X})$ is yet another Segal category $\mathcal{V}$, and finally $\mathcal{X} = l(\mathcal{V})$. In particular a single representation of a simplicial algebra, given by a derived stack $F : sk\text{-CAlg} \to \mathrm{sSet}$ comes with an identity map $id_F \in \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(\mathcal{X}, \mathcal{X})$ which is valued in $\mathcal{U}$. This means a single object F takes its full meaning only in $\mathcal{U}$, and not in $\mathcal{X}$ only. One can contrast this with the concept of having a field in Physics being defined on a target space, itself resulting from a compactification. Here the field is replaced by a functor, the target space would presumably be sSet, and having a compactified space would be replaced by having $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{SeT}}(\mathcal{X}, \mathcal{X}) = \mathcal{U}$.

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